Locally Optimal and Robust Backstepping Design

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Abstract— **For a class of nonlinear systems a new robust backstepping design achieves both local optimality and global inverse optimality. The design is robust in the sense that it achieves a prescribed level of disturbance attenuation with stability margins. An analytic example illustrates the performance of the locally optimal control design.**

Keywords— **Backstepping, local optimality, inverse optimality, robust control, disturbance attenuation**

I. Introduction

Although optimal designs, including $\mathcal{H}_\infty$-designs, are common for linear systems, optimal control of nonlinear systems is hampered by our inability to solve the Hamilton-Jacobi-Bellman (HJB) or the more general Hamilton-Jacobi-Isaacs (HJI) equation. When the nonlinear system admits a controllable linearization and the cost functional admits a Taylor series expansion with quadratic leading terms, then local approximations of the optimal control are computable [1], [2]. Although such approximations are useful, a deficiency of the existing locally optimal controllers is that they may adversely affect the system behavior in large regions of the state-space. In particular, the stability properties may deteriorate and the region of attraction may become intolerably small. Therefore, it is necessary to endow the locally optimal controllers with stronger stability properties, including a satisfactory region of attraction.

In this paper, we design locally optimal controllers which also guarantee strong global stability properties. The new design modifies the backstepping procedure of [3] and [4] so that at each step the virtual control law and the Lyapunov function are chosen to be the linear-quadratic approximations of the $\mathcal{H}_\infty$-optimal control problem, thus achieving local $\mathcal{L}_2$-disturbance attenuation. For global properties, the new design relies on inverse optimality which is known to have desirable stability margins [5], [6]. By utilizing techniques similar to those of [7], [8] and [9], the new control law is rendered globally $\mathcal{H}_\infty$-optimal in the inverse sense, that is with respect to a cost functional constructed during the recursive design procedure, and achieves global inverse $\mathcal{L}_2$-disturbance attenuation. Moreover, when $w(t) \in \mathcal{L}_\infty$, the system is rendered input-to-state stable [10]. In the absence of a disturbance, the locally LQR optimal equilibrium is globally asymptotically stable (GAS). A broad background on disturbance attenuation and inverse optimality is covered in the books [11], [12], [13], [5], while further details and extensions of the procedure presented here can be found in [14].

Two conference papers, [15] and [16], published simultaneously with an earlier version of this paper [17], contain results related to ours. In [15], it is shown that input-to-state stabilizability is both necessary and sufficient for inverse optimality of nonlinear systems, and a constructive procedure is introduced for the inverse $\mathcal{H}_\infty$-optimal design of systems in strict-feedback form. The constructive procedure in [16] designs a control law which globally achieves a finite $\mathcal{L}_2$-gain and locally matches a linear control law. Neither [15] nor [16] design controllers which are *locally* $\mathcal{H}_\infty$-optimal.

The organization of the paper is as follows. After introducing our notation in Section II and the problem formulation in Section III, we present in Section IV our new nonlinear recursive design, which starts with a linear $\mathcal{H}_\infty$-design. A detailed example in Section V illustrates the stability properties and performance of the nonlinear controller designed by the new procedure. The paper ends with some concluding remarks in Section VI.

Manuscript was submitted on November 17, 1997, and was revised on October 5, 1998. Recommended by Associate Editor, M. Krstić. This research was supported in part by the National Science Foundation under grant ECS-9528370 and the Air Force Office of Scientific Research under grant F49620-95-1-0409.

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II. Notation

For constructive methods to be developed in the sequel a simple notation is needed which uniquely specifies certain substates and subsystems that we reference during the design procedures. For example, given $A \in \mathbb{R}^{n \times n}$, the i-th linear subsystem matrix $A_{[i]}$ is defined for all $i = 1, \ldots, n$ as

$$A_{[i]} := \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} \end{bmatrix},$$

where a_{ij} is the component of A found on its ith row and jth column. The same notation is used for the subsets of states $x_{[i]}$, and vector fields, i.e., $\breve{f}_{[i]}$ and $G_{1[i]}$. The $n \times 1$ zero vector is denoted by 0_n.

Our procedures construct a convenient change of coordinates, which, for the linearized version of the system, is denoted by $z = Lx$. The same notation z is kept for the nonlinear system where $z = \Phi(x)$ is a diffeomorphism such that $\breve{\Phi}(x) := \Phi(x) - Lx$ contains only higher-order terms in x. Quantities in the transformed z-coordinates are denoted by an "over bar," such as $\bar{A}$ corresponding to A. All higher-order functions in the x-coordinates we denote with a "breve," such as $\breve{f}(x)$. Similarly, higher-order functions in the transformed coordinates are denoted by a "check," such as $\check{f}(z)$.

All absolute values and vector norms will be denoted by $|\cdot|$. For a vector norm we use $|x| := \sqrt{x'x}$. As usual, $\mathcal{L}_2$ and $\mathcal{L}_\infty$ denote the respective sets of measurable signals with finite norms

$$\|x(t)\|_2 := \left(\lim_{T\to\infty} \int_0^T x'(\tau)x(\tau)dt \right)^{1/2}$$

and

$$\|x(t)\|_\infty := \sup_{t\in[0,\infty)} \max_{1\le i\le n} |x_i(t)| .$$

III. Problem Formulation

Systems that lend themselves to backstepping designs are in strict-feedback form [3], [18], [9]

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1) + g_1(x_1)w \\ \dot{x}_2 &= x_3 + f_2(x_1,x_2) + g_2(x_1,x_2)w \\ &\vdots \\ \dot{x}_n &= f_n(x_1,\cdots,x_n) + g_n(x_1,\cdots,x_n)w + u, \end{aligned}$$

which can be written more compactly as

$$\dot{x} = f(x) + G_1(x)w + B_2 u, \tag{1}$$

where $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}'$ is the n-dimensional state; u is the scalar control input; and $w(t,x) : [0,\infty)\times \mathbb{R}^n \to \mathbb{R}^q$ is an unknown disturbance signal which is member of either $\mathcal{L}_2$ or $\mathcal{L}_\infty$. We assume that the functions f_i, g_i are sufficiently smooth and $f_i(0) = 0$, $b_i := g_i(0) \in \mathbb{R}^{1\times q}$. For these systems it is well-known how to achieve global boundedness [4], [7], [8], [9] using robust backstepping. We rely on backstepping to guarantee global boundedness, but we extend the existing backstepping procedure to also meet two additional design objectives: one local, and the other global.

Our local objective is to achieve optimality in the region where the linear dynamics dominate. For this reason we separate the linear terms:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ &\quad + \begin{bmatrix} \breve{f}_1(x_1) \\ \vdots \\ \breve{f}_n(x_1,\ldots,x_n) \end{bmatrix} + \begin{bmatrix} \breve{g}_1(x_1) \\ \vdots \\ \breve{g}_n(x_1,\ldots,x_n) \end{bmatrix} w \\ &= Ax + B_1 w + B_2 u + \breve{f}(x) + \breve{G}_1(x)w, \end{aligned}$$

where $\breve{f}(x) = f(x) - Ax$, $B_1 = G_1(0)$, $\breve{G}_1(x) = G_1(x) - B_1$,

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} a_{11} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 1 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix},$$

and $B_2 = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}'$. The pair (A, B_2) is controllable irrespective of the values of the coefficients a_{ij}. To meet our first objective we wish to find a locally stabilizing control law $u_l = \mu_l(x)$, which minimizes the $\mathcal{L}_2$-gain from the disturbance w to the controlled output $\mathfrak{z}_l = C_1 x + D_1 u_l$ where $Q := C_1' C_1$, $R := D_1' D_1$, and $D_1' C_1 = 0$. As shown in [12], this disturbance attenuation problem is equivalent to the *dynamic game* $\min_{u_l} \max_{w_l} J_l(u_l, w_l)$ for the linearized system dynamics $\dot{x} = Ax + B_1 w_l + B_2 u_l$ and

$$J_l(u_l, w_l) = \int_0^\infty [x'Qx + Ru_l^2 - \gamma^2 w_l' w_l] dt, \tag{2}$$

where $Q = Q' > 0$ and $R > 0$. The optimal attenuation level $\gamma^* > 0$ exists because (A, B_2) is controllable and (A, Q) is observable. The desired attenuation level is a fixed $\gamma > \gamma^*$.

Our global objective is to achieve global inverse optimality with respect to the cost functional

$$J(u, w) = \int_0^\infty \left[q(x) + r(x)u^2 - \gamma^2 w'w \right] dt. \tag{3}$$

Our approach is inverse because we do not specify $q(x)$ and $r(x)$ beforehand. However, to assure that the cost (3) is *meaningful*, we require that

$$q(x) \text{ is positive definite, and } r(x) > 0 \;\; \forall x. \tag{4}$$

Furthermore, to satisfy the local optimality objective we impose the requirements[1]

$$r(0) = R \qquad \text{and} \qquad q_{xx}(0) := \frac{1}{2} \left.\frac{\partial^2 q(x)}{\partial x^2}\right|_{x=0} = Q. \tag{5}$$

We meet our local and global objectives with a significant extension of the backstepping procedure. At the i-th step of backstepping, we recursively construct a Lyapunov function $V_i(x_{[i]})$ and determine the i-th virtual control $\alpha_i(x_{[i]})$ such that if $x_{i+1} \equiv \alpha_i(x_{[i]})$

$$\dot{V}_i = -x_{[i]}' Q_{T[i]} x_{[i]} + \gamma^2 w'w - \gamma^2 \left| w - \nu_i(x_{[i]}) \right|^2, \tag{6}$$

where $x_{[i]} := \begin{bmatrix} x_1 & \ldots & x_i \end{bmatrix}'$ and $Q_{T[i]}$ is a positive definite matrix whose dependence on the matrix Q will be depicted later. The i-th worst-case disturbance is

$$\nu_i(x_{[i]}) := \frac{1}{2\gamma^2} G_{1[i]}'(x_{[i]}) \frac{\partial V_i'}{\partial x_{[i]}}$$

with $G_{1[i]} := \begin{bmatrix} g_1' & \cdots & g_i' \end{bmatrix}'$. The dissipation inequality

$$\dot{V}_i \le -x_{[i]}' Q_{T[i]} x_{[i]} + \gamma^2 w'w$$

follows directly from (6). By construction, the final Lyapunov function $V(x) > 0$, obtained at $i = n$, is the value function of the dynamic game $\min_u \max_w J(u, w)$ for the cost (3) constrained by system (1). For a desired level of attenuation $\gamma > \gamma^*$, the satisfaction of the HJI equation

$$\begin{aligned} V_x(x)f(x) + \frac{1}{4} V_x(x) \left(\gamma^{-2} G_1(x) G_1'(x) \right. \\ \left. - B_2 r^{-1}(x) B_2' \right) V_x'(x) + q(x) = 0 \end{aligned}$$

[1] This definition of $q_{xx}(0)$ differs from the Hessian because of the additional factor of $1/2$ which is included for convenience.

is achieved by selecting $q(x)$ and $r(x)$ which meet the positivity (4) and local optimality (5) requirements. The resulting optimal control law

$$u = \mu(x) := -\frac{1}{2} r^{-1}(x) B_2' V_x'(x), \tag{7}$$

and worst-case disturbance $w = \nu(x) := \dfrac{1}{2\gamma^2} G'(x) V_x'(x)$, satisfy

$$\min_u \max_w \left[q(x) + r(x)u^2 - \gamma^2 w'w + \dot V(x) \right] = 0$$

and the dissipation inequality

$$\begin{aligned} \dot V &= -q(x) - r(x)u^2 + \gamma^2 w'w \\ &\quad -\gamma^2 |w-\nu|^2 + r(x)(u-\mu)^2 \\ &\leq -q(x) - r(x)u^2 + \gamma^2 w'w. \end{aligned}$$

Thus, the $\mathcal{L}_2$-gain from the disturbance w to the controlled output is bounded from above by γ. The controlled output, sometimes called the regulation output, is defined as $\mathfrak{z} = H_{11}(x)x + H_{12}(x)u$ such that $q(x) = x'H_{11}'(x)H_{11}(x)x$, $r(x) = H_{12}'(x)H_{12}(x)$, and $H_{12}'(x)H_{11}(x) = 0$. Furthermore, $C_1 \equiv H_{11}(0)$ and $D_1 \equiv H_{12}(0)$.

IV. Locally Optimal Backstepping Design

The new locally optimal backstepping design is presented in four parts. First, we introduce a factorization of the solution to the *generalized algebraic Riccati equation* (GARE) for the linear $\mathcal{H}_\infty$-design. This factorization is instrumental in a recursive backstepping procedure resulting in the same linear control law as in the linear $\mathcal{H}_\infty$-design. We then augment this recursive backstepping procedure to incorporate the remaining nonlinearities at each step. Finally, we construct the state and control penalties $q(x)$ and $r(x)$, and the control law which achieves global inverse optimality.

A. Factorization of the Linear $\mathcal{H}_\infty$-Design

The objective of the $\mathcal{H}_\infty$-design for the linear part of system (1),

$$\dot x = Ax + B_1 w_l + B_2 u_l, \tag{8}$$

is to find the controller which minimizes the cost (2) for the worst-case disturbance. We are assured that the required unique positive definite solution $P = P' > 0$ of the GARE

$$0 = PA + A'P + P\left(\frac{1}{\gamma^2} B_1 B_1' - B_2 R^{-1} B_2'\right) P + Q \tag{9}$$

exists for all $\gamma > \gamma^* > 0$ because (A, B_2) is controllable and (A, Q) is observable [12]. The worst-case optimal control law for the cost functional (2) is

$$u_l = \mu_l(x) := -R^{-1} B_2' P x, \tag{10}$$

and the corresponding worst-case disturbance is $w_l = \nu_l(x) := \dfrac{1}{\gamma^2} B_1' P x$.

A factorization of P which will make the $\mathcal{H}_\infty$-design suitable for backstepping is $P = L'\Delta L$, where L and Δ are

$$\begin{aligned} L &:= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha_{11} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\alpha_{n-1,1} & \cdots & -\alpha_{n-1,n-1} & 1 \end{bmatrix}, \\ \Delta &:= \operatorname{diag}(\delta_1, \ldots, \delta_n), \end{aligned} \tag{11}$$

with $\delta_i > 0$ for $i = 1, \ldots, n$. It is a matter of simple matrix algebra to show that $\Delta^{1/2} L = U'^{-1}$ where U is the unique Cholesky factor of $P^{-1} = U'U$. Therefore, the above pair (Δ, L) is unique. Furthermore, under the state transformation $z = Lx$, the matrix L has the following property.

Property 1: For $1 \le k \le n$: $z_{[k]} = L_{[k]} x_{[k]}$ where $L_{[k]}$ is invertible. ◇

Applying the transformation $z = Lx$ to the linear system (8) we obtain

$$\dot z = \bar A z + \bar B_1 w_l + B_2 u_l, \quad \bar A = LAL^{-1}, \quad \bar B_1 = LB_1. \tag{12}$$

Due to the structure of L, the structure of $\bar A$ is identical to that of A.

Property 2: For $1 \le k < n$: $\dot z_{[k]} = \bar A_{[k]} z_{[k]} + \begin{bmatrix} 0 \\ \vdots \\ z_{k+1} \end{bmatrix} + \bar B_{1[k]} w_l$. ◇

In the GARE (9) we replace P with $L'\Delta L$ and, upon the pre-multiplication by L'^{-1} and post-multiplication by L^{-1}, we obtain

$$0 = \Delta \bar A + \bar A' \Delta + \Delta\left(\frac{1}{\gamma^2} \bar B_1 \bar B_1' - B_2 R^{-1} B_2'\right)\Delta + \bar Q, \tag{13}$$

where $\bar Q = L'^{-1} Q L^{-1} > 0$. In this way we discover an important property of the transformed GARE (13).

Property 3: For $1 \le k < n$: $0 = \bar A_{[k]}' \Delta_{[k]} + \Delta_{[k]} \bar A_{[k]} + \dfrac{1}{\gamma^2} \Delta_{[k]} \bar B_{1[k]} \bar B_{1[k]}' \Delta_{[k]} + \bar Q_{[k]}$. ◇

A consequence of Property 3 is that $\bar A_{[k]}$ is Hurwitz for $k = 1, \ldots, n-1$. Furthermore, for all k such that $1 \le k < n$,

$$2 z_{[k]}' \Delta_{[k]} \bar A_{[k]} z_{[k]} = -z_{[k]}' \bar Q_{[k]} z_{[k]} - \gamma^2 \bar\nu_{lk}' \bar\nu_{lk}, \tag{14}$$

where $\bar\nu_{lk}(z_{[k]}) := \dfrac{1}{\gamma^2} \bar B_{1[k]}' \Delta_{[k]} z_{[k]}$.

The transformed GARE (13) implies that $\bar V = z'\Delta z$ is the value function of the game

$$\min_{u_l} \max_{w_l} J_l(u_l, w_l)$$

for the linear system (12) and cost functional

$$J_l(u_l, w_l) = \int_0^\infty [z'\bar Q z + R u_l^2 - \gamma^2 w_l' w_l] dt. \tag{15}$$

The function $\bar{V}(z)$ can also be considered as a Lyapunov function and as a storage function. To derive the underlying dissipation inequality, we add $\gamma^2 w_l' w_l - \gamma^2 w_l' w_l + R u_l^2 - R u_l^2 \equiv 0$ to the derivative of $\bar{V}$, complete the squares with respect to both u_l and w_l, to obtain

$$\begin{aligned} \dot{\bar{V}} &= -z'\bar{Q}z - Ru_l^2 + \gamma^2 w_l' w_l \\ &\quad -\gamma^2 \left|w_l - \bar{\nu}_l\right|^2 + R\left(u_l - \bar{\mu}_l\right)^2 \\ &\le -z'\bar{Q}z - Ru_l^2 + \gamma^2 w_l' w_l, \end{aligned}$$

where the first equality is arrived at by utilizing the algebraic relationship (13), and the last inequality is guaranteed by the control law $u_l = \bar{\mu}_l(z) := -R^{-1}B_2'\Delta z$. The corresponding worst-case disturbance is given by $\bar{\nu}_l(z) := \frac{1}{\gamma^2}\bar{B}_1'\Delta z$.

B. Linear Backstepping: Local Optimality

We now describe a linear backstepping procedure which is preparatory for the more difficult nonlinear design. In linear backstepping we restrict ourselves to the linear part (8) of the nonlinear system (1). We construct linear virtual controllers of the form $\alpha_i(x_{[i]}) = \alpha_{[i]}x_{[i]}$, where the $\alpha_{[i]}$'s are defined as

$$\alpha_{[i]} := \begin{bmatrix}\alpha_{i1} & \cdots & \alpha_{ii}\end{bmatrix} \qquad i = 1, \ldots, n-1,$$

with $\alpha_{[0]} = 0$ and $\alpha_{[n]} := 0_n'$. Here the $\{\alpha_{ij}\}$'s are the elements of the matrix L specified in (11). Next we introduce the "error coordinates"

$$\begin{aligned} z_1 &= x_1 \\ z_i &= x_i - \alpha_{[i-1]}x_{[i-1]} \qquad i = 2, \ldots, n, \end{aligned}$$

given by the transformation $z = Lx$. At each step i of the backstepping procedure we select the positive definite Lyapunov function

$$\bar{V}_i(z_{[i]}) = \bar{V}_{i-1}(z_{[i-1]}) + \delta_i z_i^2,\ \bar{V}_0 := 0,\ i = 1, \ldots, n.$$

At step $i = n$ we get $\bar{V} = z'\Delta z$, the solution of the HJI equation corresponding to the GARE (13). To describe the steps of this recursive procedure we first rewrite system (8) as

$$\begin{aligned} \dot{x}_i &= a_{[i]}x_{[i]} + x_{i+1} + b_i w_l \qquad i = 1, \ldots, n-1, \\ \dot{x}_n &= a_{[n]}x_{[n]} + b_n w_l + u_l, \end{aligned}$$

where $a_{[i]} := \begin{bmatrix}a_{i1} & \ldots & a_{ii}\end{bmatrix}$. We also define the row vector $\bar{\alpha}_{[i]}$ as

$$\bar{\alpha}_{[i]} := \alpha_{[i]}L_{[i]}^{-1} \qquad i = 1, \ldots, n,$$

noting that $L_{[i]}^{-1} := (L_{[i]})^{-1}$.

Step 1: Define $z_1 = x_1$. Select $\bar{\alpha}_1(z_{[1]}) = \bar{\alpha}_{[1]}z_{[1]}$ as the virtual control law for x_2, and $\bar{V}_1 = z_{[1]}'\Delta_{[1]}z_{[1]}$ as its value function. Since $L_{[1]} = 1$, the dynamics of z_1 are

$$\dot{z}_1 = (a_{[1]} + \bar{\alpha}_{[1]})z_{[1]} + (x_2 - \bar{\alpha}_1) + b_1 w_l.$$

From Property 2 for $k = 1$, we see that the z_1-subsystem can be written as

$$\dot{z}_1 = \bar{a}_{[1]}z_{[1]} + z_2 + \bar{b}_1 w_l, \quad \bar{a}_{[1]} = a_{[1]} + \bar{\alpha}_{[1]}, \quad \bar{b}_1 = b_1.$$

Adding $\gamma^2 w_l' w_l - \gamma^2 w_l' w_l = 0$ to the derivative of $\bar{V}_1$, completing the squares with respect to w_l, and using (14) we obtain

$$\dot{\bar{V}}_1 = -z_{[1]}'\bar{Q}_{[1]}z_{[1]} + \gamma^2 w_l' w_l - \gamma^2\left|w_l - \bar{\nu}_{l1}\right|^2 + 2z_1\delta_1 z_2,$$

where $\bar{\nu}_{l1}(z_{[1]}) := \frac{1}{\gamma^2}\bar{b}_1'\delta_1 z_1 = \frac{1}{\gamma^2}\bar{B}_{1[1]}'\Delta_{[1]}z_{[1]}$. If $z_2 \equiv 0$, then $\bar{V}_1 \ge 0$ is a storage function with supply rate $-z_{[1]}'\bar{Q}_{[1]}z_{[1]} + \gamma^2 w_l' w_l$. Therefore, by the results of [19], we have $z_1(t) \to 0$ as $t \to \infty$ for all $w(t) \in \mathcal{L}_2$, and $z_1(t)$ is bounded for all $w(t) \in \mathcal{L}_\infty$. In the absence of a disturbance $z_1 = 0$ is GAS.

Step i: Define $z_i = x_i - \bar{\alpha}_{i-1}(z_{[i-1]})$ and select the value function for this step to be $\bar{V}_i = \bar{V}_{i-1} + \delta_i z_i^2 = z_{[i]}'\Delta_{[i]}z_{[i]}$. From the preceding step, the dynamics of the $z_{[i-1]}$-subsystem are described by Property 2 for $k = i-1$. The derivative of $\bar{V}_{i-1}$ is expressed as

$$\begin{aligned} \dot{\bar{V}}_{i-1} &= -z_{[i-1]}'\bar{Q}_{[i-1]}z_{[i-1]} + \gamma^2 w_l' w_l \\ &\quad -\gamma^2\left|w_l - \bar{\nu}_{l,i-1}\right|^2 + 2z_{i-1}\delta_{i-1}z_i, \end{aligned}$$

where $\bar{\nu}_{l,i-1}(z_{[i-1]}) := \frac{1}{\gamma^2}\bar{B}_{1[i-1]}'\Delta_{[i-1]}z_{[i-1]}$. The dynamics of z_i are given by

$$\begin{aligned} \dot{z}_i &= a_{[i]}L_{[i]}^{-1}z_{[i]} + x_{i+1} + b_i w_l \\ &\quad -\frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\left[\bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix}0_{i-2}\\ z_i\end{bmatrix} + \bar{B}_{1[i-1]}w_l\right]. \end{aligned}$$

Selecting for x_{i+1} the virtual control law $\bar{\alpha}_i(z_{[i]}) = \bar{\alpha}_{[i]}z_{[i]}$ we get

$$\dot{z}_i = \bar{a}_{[i]}z_{[i]} + z_{i+1} + \bar{b}_i w_l,$$

where we have used Properties 1 and 2 to establish the identities

$$\bar{a}_{[i]} = a_{[i]}L_{[i]}^{-1} + \bar{\alpha}_{[i]} - \begin{bmatrix}\bar{\alpha}_{[i-1]}\bar{A}_{[i-1]} & \bar{\alpha}_{i-1,i-1}\end{bmatrix} \tag{16a}$$

$$\bar{b}_i = b_i - \bar{\alpha}_{[i-1]}\bar{B}_{1[i-1]}. \tag{16b}$$

The term $\bar{\alpha}_{i-1,i-1}$ is defined as the $(i-1)$st element of $\bar{\alpha}_{[i-1]}$. The derivative of $\bar{V}_i$ at this step is

$$\dot{\bar{V}}_i = -z_{[i]}'\bar{Q}_{[i]}z_{[i]} + \gamma^2 w_l' w_l - \gamma^2\left|w_l - \bar{\nu}_{li}\right|^2 + 2z_i\delta_i z_{i+1},$$

where

$$\bar{\nu}_{li}(z_{[i]}) := \bar{\nu}_{l,i-1} + \frac{1}{\gamma^2}\bar{b}_i'\delta_i z_i = \frac{1}{\gamma^2}\bar{B}_{1[i]}'\Delta_{[i]}z_{[i]}$$

and the dynamics of the $z_{[i]}$-subsystem are described by Property 2 for $k = i$. When $z_{i+1} \equiv 0$, $\bar{V}_i \ge 0$ is a storage function with supply rate $-z_{[i]}'\bar{Q}_{[i]}z_{[i]} + \gamma^2 w_l' w_l$. By the results of [19], for all $w(t) \in \mathcal{L}_2$, we have $z_{[i]}(t) \to 0$ as $t \to \infty$, and $z_{[i]}(t)$ is bounded for all $w(t) \in \mathcal{L}_\infty$. In the absence of a disturbance, the $z_{[i]}$-subsystem is GAS.

Step n: At the final step of the linear backstepping procedure, we define $z_n = x_n - \bar{\alpha}_{n-1}(z_{[n-1]})$ and choose

$$\bar{V} = \bar{V}_{n-1} + \delta_n z_n^2 = z'\Delta z$$

as the value function. The dynamics of z_n are given by

$$\begin{aligned} \dot{z}_n &= a_{[n]}L^{-1}z + b_n w_l + u_l \\ &\quad - \frac{\partial \bar{\alpha}_{n-1}}{\partial z_{[n-1]}} \left[\bar{A}_{[n-1]} z_{[n-1]} + \begin{bmatrix} 0_{n-2} \\ z_n \end{bmatrix} + \bar{B}_{1[n-1]} w_l \right]. \end{aligned}$$

Using the identity (16) with $i = n$, we rewrite the z_n-dynamics as $\dot{z}_n = \bar{a}_{[n]}z + \bar{b}_n w_l + u_l$. Therefore, the transformed system is now described by (12). Finally, after a completion of squares with respect to both u_l and w_l, we get

$$\begin{aligned} \dot{\bar{V}} &= -z'\bar{Q}z - Ru_l^2 + \gamma^2 w_l' w_l \\ &\quad -\gamma^2 |w_l - \bar{\nu}_l|^2 + R(u_l - \bar{\mu}_l)^2, \end{aligned}$$

where $\bar{\nu}_l(z) := \bar{\nu}_{l,n-1} + \frac{1}{\gamma^2}\bar{b}_n' \delta_n z_n = \frac{1}{\gamma^2}\bar{B}_1'\Delta z$ is the worst-case disturbance. The control law

$$\bar{\mu}_l(z) := -R^{-1}B'\Delta z \tag{17}$$

is the $\mathcal{H}_\infty$-controller (10) in the z-coordinates. When $u_l = \bar{\mu}_l(z)$, we obtain

$$\begin{aligned} \dot{\bar{V}} &= -z'\bar{Q}z - Ru_l^2 + \gamma^2 w_l' w_l - \gamma^2 |w_l - \bar{\nu}_l|^2 \\ &\leq -z'\bar{Q}z - Ru_l^2 + \gamma^2 w_l' w_l. \end{aligned}$$

By this construction, $\bar{V}(z)$ is the solution to the HJI equation corresponding to the GARE (13). We have therefore successfully completed the linear backstepping procedure which stabilizes system (8) while minimizing the cost functional (2) for the worst-case disturbance $\bar{\nu}_l(z)$.

C. Nonlinear Backstepping: Global Properties

The nonlinear backstepping design procedure builds upon the linear procedure described in the previous section. At each step of the design we select as our virtual control the same terms as in the linear case and add a term to cancel the nonlinearities and to attenuate the disturbance. The cancellations are only employed to simplify the presentation. The procedure can be modified to avoid canceling beneficial nonlinearities. With the completion of each step, a new state $z_i = \phi_i(x_{[i]})$ is defined by our choice of the virtual control law from the previous step. At the last step, this construction results in the lower-triangular diffeomorphism $z = \Phi(x)$ and control Lyapunov function (CLF) $\bar{V}(z)$.

Step 1: Define $z_1 = \phi_1(x_1) := x_1$ and select $\bar{V}_1 = z_{[1]}'\Delta_{[1]}z_{[1]} > 0$ as the value function. Since $x_{[1]} = \Phi_{[1]}^{-1}(z_{[1]}) = z_{[1]}$, the $z_{[1]}$-subsystem is

$$\dot{z}_1 = a_{[1]}z_{[1]} + \breve{f}_1(z_{[1]}) + x_2 + g_1(z_{[1]})w.$$

The recursive design of the nonlinear control law begins by adding a nonlinear term $\breve{\alpha}_1(z_{[1]})$ to the linear virtual control law for x_2, that is $\bar{\alpha}_1(z_{[1]}) = \bar{\alpha}_{[1]}z_{[1]} + \breve{\alpha}_1(z_{[1]})$. Using (16) with $i = 1$ we get

$$\begin{aligned} \dot{z}_1 &= \bar{a}_{[1]}z_{[1]} + \check{f}_1(z_{[1]}) + (x_2 - \bar{\alpha}_1(z_{[1]})) + \bar{g}_1(z_{[1]})w \\ \dot{\bar{V}}_1 &= 2z_{[1]}'\Delta_{[1]}\left(\bar{a}_{[1]}z_{[1]} + \check{f}_1 + (x_2 - \bar{\alpha}_1) + \bar{g}_1 w\right), \end{aligned}$$

where $\check{f}_1(z_{[1]}) := \breve{\alpha}_1(z_{[1]}) + \breve{f}_1(z_{[1]})$, and $\bar{g}_1(z_{[1]}) := g_1(z_{[1]})$. Adding $\gamma^2 w'w - \gamma^2 w'w \equiv 0$ to $\dot{\bar{V}}_1$, completing the squares with respect to w, and using (14) with $k = 1$, we find

$$\begin{aligned} \dot{\bar{V}}_1 &= -z_{[1]}'\bar{Q}_{[1]}z_{[1]} + \gamma^2 w'w - \gamma^2 |w - \bar{\nu}_1|^2 \\ &\quad + 2z_{[1]}'\Delta_{[1]}\left[\check{f}_1 + \frac{1}{2\gamma^2}\left(\bar{g}_1\bar{g}_1' - \bar{b}_1\bar{b}_1'\right)\Delta_{[1]}z_{[1]}\right] \\ &\quad + 2z_1\delta_1(x_2 - \bar{\alpha}_1), \end{aligned}$$

where $\bar{\nu}_1(z_{[1]}) := \frac{1}{\gamma^2}\bar{g}_1'(z_1)\delta_1 z_1 = \frac{1}{\gamma^2}\bar{G}_{1[1]}'(z_{[1]})\Delta_{[1]}z_{[1]}$. Now we select $\breve{\alpha}_1$ to cancel the term in the brackets:

$$\begin{aligned} \breve{\alpha}_1 &= -\breve{f}_1 - \frac{1}{2\gamma^2}\left(\bar{g}_1\bar{g}_1' - \bar{b}_1\bar{b}_1'\right)\Delta_{[1]}z_{[1]} \\ &\Rightarrow \quad \check{f}_1 = -\frac{1}{2\gamma^2}\left(\bar{g}_1\bar{g}_1' - \bar{b}_1\bar{b}_1'\right)\Delta_{[1]}z_{[1]}, \end{aligned}$$

which brings $\dot{\bar{V}}_1$ to the desired dissipativity form

$$\begin{aligned} \dot{\bar{V}}_1 &= -z_{[1]}'\bar{Q}_{[1]}z_{[1]} + \gamma^2 w'w - \gamma^2 |w - \bar{\nu}_1|^2 \\ &\quad + 2z_1\delta_1(x_2 - \bar{\alpha}_1) \\ &= -z_{[1]}'\bar{Q}_{[1]}z_{[1]} - \gamma^2\bar{\nu}_1'\bar{\nu}_1 + 2\gamma^2\bar{\nu}_1'w \\ &\quad + 2z_1\delta_1(x_2 - \bar{\alpha}_1). \end{aligned}$$

Indeed, the dissipation inequality $\dot{\bar{V}}_1 \leq -z_{[1]}'\bar{Q}_{[1]}z_{[1]} + \gamma^2 w'w$ is satisfied when $x_2 \equiv \bar{\alpha}_1(z_{[1]})$. Therefore, $\bar{V}_1 \geq 0$ is a storage function with supply rate $-z_{[1]}'\bar{Q}_{[1]}z_{[1]} + \gamma^2 w'w$ and the $z_{[1]}$-subsystem has been stabilized. Furthermore, for all $w(t) \in \mathcal{L}_2$, we have $z_{[1]}(t) \to 0$ as $t \to \infty$ [19]. In the absence of a disturbance, the equilibrium $z_{[1]} = 0$ of the first nonlinear subsystem is GAS.

Step i: Define $z_i = \phi_i(x_{[i]}) := x_i - \bar{\alpha}_{i-1}(z_{[i-1]})$ where $\bar{\alpha}_{i-1}(z_{[i-1]}) = \bar{\alpha}_{[i-1]}z_{[i-1]} + \breve{\alpha}_{i-1}(z_{[i-1]})$, and select the value function $\bar{V}_i = \bar{V}_{i-1} + \delta_i z_i^2 = z_{[i]}'\Delta_{[i]}z_{[i]} > 0$. In the recursion, we assume that the preceding step results in

$$\begin{aligned} \dot{z}_{[i-1]} &= \bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix} 0_{i-2} \\ z_i \end{bmatrix} + \check{f}_{[i-1]}(z_{[i-1]}) \\ &\quad + \bar{G}_{1[i-1]}(z_{[i-1]})w, \end{aligned}$$

and

$$\begin{aligned} \dot{\bar{V}}_{i-1} &= -z_{[i-1]}'\bar{Q}_{[i-1]}z_{[i-1]} + \gamma^2 w'w - \gamma^2 |w - \bar{\nu}_{i-1}|^2 \\ &\quad + 2z_{i-1}\delta_{i-1}z_i \\ &= -z_{[i-1]}'\bar{Q}_{[i-1]}z_{[i-1]} - \gamma^2\bar{\nu}_{i-1}'\bar{\nu}_{i-1} + 2\gamma^2\bar{\nu}_{i-1}'w \\ &\quad + 2z_{i-1}\delta_{i-1}z_i \\ &= 2z_{[i-1]}'\Delta_{[i-1]}\bar{A}_{[i-1]}z_{[i-1]} + \gamma^2\bar{\nu}_{l,i-1}'\bar{\nu}_{l,i-1} \\ &\quad - \gamma^2\bar{\nu}_{i-1}'\bar{\nu}_{i-1} + 2\gamma^2\bar{\nu}_{i-1}'w + 2z_{i-1}\delta_{i-1}z_i, \end{aligned}$$

where $\bar{\nu}_{i-1}(z_{[i-1]}) := \frac{1}{\gamma^2}\bar{G}'_{1[i-1]}(z_{[i-1]})\Delta_{[i-1]}z_{[i-1]}$. The z_i-dynamics are given by

$$\begin{aligned}\dot{z}_i &= a_{[i]}\Phi_{[i]}^{-1}(z_{[i]}) + \breve{f}_i(\Phi_{[i]}^{-1}(z_{[i]})) + x_{i+1}\\ &\quad + g_i(\Phi_{[i]}^{-1}(z_{[i]}))w - \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\Bigg[\bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix}0_{i-2}\\ z_i\end{bmatrix}\\ &\qquad + \breve{f}_{[i-1]}(z_{[i-1]}) + \bar{G}_{1[i-1]}(z_{[i-1]})w\Bigg],\end{aligned}$$

where we have used $x_{[i]} = \Phi_{[i]}^{-1}(z_{[i]})$. We now select $\bar{\alpha}_i(z_{[i]}) = \bar{a}_{[i]}z_{[i]} + \breve{\alpha}_i(z_{[i]})$ as the virtual control for x_{i+1} and, using (16), obtain the z_i-dynamics

$$\dot{z}_i = \bar{a}_{[i]}z_{[i]} + \breve{f}_i(z_{[i]}) + (x_{i+1} - \bar{\alpha}_i(z_{[i]})) + \bar{g}_i(z_{[i]})w,$$

where

$$\begin{aligned}\breve{f}_i(z_{[i]}) &:= \breve{\alpha}_i(z_{[i]}) + a_{[i]}\breve{\Psi}_{[i]}(z_{[i]}) + \breve{f}_i(\Phi_{[i]}^{-1}(z_{[i]}))\\ &\quad - \frac{\partial\breve{\alpha}_{i-1}}{\partial z_{[i-1]}}\left[\bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix}0_{i-2}\\ z_i\end{bmatrix}\right]\\ &\quad - \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\breve{f}_{[i-1]}(z_{[i-1]})\\ \bar{g}_i(z_{[i]}) &:= g_i(\Phi_{[i]}^{-1}(z_{[i]})) - \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\bar{G}_{1[i-1]}(z_{[i-1]}),\end{aligned}$$

and $\breve{\Psi}_{[i]}(z_{[i]}) := \Phi_{[i]}^{-1}(z_{[i]}) - L_{[i]}^{-1}z_{[i]}$ contains only higher-order terms. Therefore, the $z_{[i]}$-dynamics is given by

$$\dot{z}_{[i]} = \bar{A}_{[i]}z_{[i]} + \breve{f}_{[i]}(z_{[i]}) + \bar{G}_{1[i]}(z_{[i]})w + \begin{bmatrix}0_{i-1}\\ x_{i+1} - \bar{\alpha}_i\end{bmatrix}.$$

Using the identity

$$\begin{aligned}z'_{[i]}\Delta_{[i]}\bar{A}_{[i]}z_{[i]} &= z'_{[i-1]}\Delta_{[i-1]}\bar{A}_{[i-1]}z_{[i-1]}\\ &\quad + z_{i-1}\delta_{i-1}z_i + z_i\delta_i\bar{a}_{[i]}z_{[i]},\end{aligned}$$

we obtain

$$\begin{aligned}\dot{\bar{V}}_i &= 2z'_{[i]}\Delta_{[i]}\bar{A}_{[i]}z_{[i]} + \gamma^2\bar{\nu}'_{l,i-1}\bar{\nu}_{l,i-1} - \gamma^2\bar{\nu}'_{i-1}\bar{\nu}_{i-1}\\ &\quad + 2\gamma^2\bar{\nu}'_i w + 2z_i\delta_i\left(\breve{f}_i + (x_{i+1} - \bar{\alpha}_i)\right),\end{aligned}\tag{18}$$

where $\bar{\nu}_i(z_{[i]}) := \bar{\nu}_{i-1} + \frac{1}{\gamma^2}\bar{g}'_i\delta_i z_i = \frac{1}{\gamma^2}\bar{G}'_{1[i]}(z_{[i]})\Delta_{[i]}z_{[i]}$. Using the GARE identity (14),

$$\begin{aligned}\dot{\bar{V}}_i &= -z'_{[i]}\bar{Q}_{[i]}z_{[i]} - \gamma^2\left(\bar{\nu}'_{li}\bar{\nu}_{li} - \bar{\nu}'_{l,i-1}\bar{\nu}_{l,i-1}\right)\\ &\quad - \gamma^2\bar{\nu}'_{i-1}\bar{\nu}_{i-1} + 2\gamma^2\bar{\nu}'_i w + 2z_i\delta_i\left(\breve{f}_i + x_{i+1} - \bar{\alpha}_i\right).\end{aligned}$$

We now add $\gamma^2 w'w - \gamma^2 w'w \equiv 0$, and complete the squares with respect to w,

$$\begin{aligned}\dot{\bar{V}}_i &= -z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w - \gamma^2|w - \bar{\nu}_i|^2\\ &\quad + \gamma^2\left(\bar{\nu}'_i\bar{\nu}_i - \bar{\nu}'_{i-1}\bar{\nu}_{i-1}\right) - \gamma^2\left(\bar{\nu}'_{li}\bar{\nu}_{li} - \bar{\nu}'_{l,i-1}\bar{\nu}_{l,i-1}\right)\\ &\quad + 2z_i\delta_i\breve{f}_i + 2z_i\delta_i(x_{i+1} - \bar{\alpha}_i)\\ &= -z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w - \gamma^2|w - \bar{\nu}_i|^2\\ &\quad + 2z_i\delta_i[\breve{f}_i + \left(\bar{g}_i\bar{\nu}_{i-1} - \bar{b}_i\bar{\nu}_{l,i-1}\right)\\ &\quad + \frac{1}{2\gamma^2}\left(\bar{g}_i\bar{g}'_i - \bar{b}_i\bar{b}'_i\right)\delta_i z_i] + 2z_i\delta_i(x_{i+1} - \bar{\alpha}_i).\end{aligned}$$

Then we select $\breve{\alpha}_i$ to cancel the term in the brackets:

$$\begin{aligned}\breve{\alpha}_i(z_{[i]}) &= -a_{[i]}\breve{\Psi}_{[i]}(z_{[i]}) - \breve{f}_i(\Phi_{[i]}^{-1}(z_{[i]}))\\ &\quad + \frac{\partial\breve{\alpha}_{i-1}}{\partial z_{[i-1]}}\left[\bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix}0_{i-2}\\ z_i\end{bmatrix}\right]\\ &\quad + \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\breve{f}_{[i-1]}(z_{[i-1]}) - \left(\bar{g}_i\bar{\nu}_{i-1} - \bar{b}_i\bar{\nu}_{l,i-1}\right)\\ &\quad - \frac{1}{2\gamma^2}\left(\bar{g}_i\bar{g}'_i - \bar{b}_i\bar{b}'_i\right)\delta_i z_i\\ \Rightarrow \breve{f}_i &= -\left(\bar{g}_i\bar{\nu}_{i-1} - \bar{b}_i\bar{\nu}_{l,i-1}\right) - \frac{1}{2\gamma^2}\left(\bar{g}_i\bar{g}'_i - \bar{b}_i\bar{b}'_i\right)\delta_i z_i.\end{aligned}$$

This results in the dissipative form of $\dot{\bar{V}}_i$:

$$\begin{aligned}\dot{\bar{V}}_i &= -z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w - \gamma^2|w - \bar{\nu}_i|^2\\ &\quad + 2z_i\delta_i(x_{i+1} - \bar{\alpha}_i)\\ &\le -z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w + 2z_i\delta_i(x_{i+1} - \bar{\alpha}_i).\end{aligned}$$

Indeed, when $x_{i+1} \equiv \bar{\alpha}_i(z_{[i]})$, $\bar{V}_i \ge 0$ is a storage function with supply rate $-z'_{[i]}\bar{Q}_{[i]}z_{[i]} + \gamma^2 w'w$. Therefore, $z_{[i]}(t)$ is globally bounded for all $w(t) \in \mathcal{L}_\infty$, and for all $w(t) \in \mathcal{L}_2$, we have $z_{[i]}(t) \to 0$ as $t \to \infty$. In the absence of a disturbance, the equilibrium $z_{[i]} = 0$ is GAS.

Step n: For the final step we define $z_n = \phi_n(x) := x_n - \bar{\alpha}_{n-1}(z_{[n-1]})$, and select $\bar{V} = \bar{V}_{n-1} + \delta_n z_n^2 = z'\Delta z > 0$. Then

$$\begin{aligned}\dot{z}_n &= a_{[n]}\Phi^{-1}(z) + \breve{f}_n(\Phi^{-1}(z)) + u + g_n(\Phi^{-1}(z))w\\ &\quad - \frac{\partial\bar{\alpha}_{n-1}}{\partial z_{[n-1]}}\Bigg[\bar{A}_{[n-1]}z_{[n-1]} + \begin{bmatrix}0_{n-2}\\ z_n\end{bmatrix}\\ &\qquad + \breve{f}_{[n-1]}(z_{[n-1]}) + \bar{G}_{1[n-1]}(z_{[n-1]})w\Bigg],\end{aligned}$$

which can be rewritten as

$$\dot{z}_n = \bar{a}_{[n]}z + \breve{f}_n(z) + \bar{g}_n(z)w + u,$$

where $\bar{a}_{[n]}$ is defined by formula (16) for $i = n$, and

$$\begin{aligned}\breve{f}_n(z) &:= a_{[n]}\breve{\Psi}(z) + \breve{f}_n(\Phi^{-1}(z))\\ &\quad - \frac{\partial\breve{\alpha}_{n-1}}{\partial z_{[n-1]}}\left[\bar{A}_{[n-1]}z_{[n-1]} + \begin{bmatrix}0_{n-2}\\ z_n\end{bmatrix}\right]\\ &\quad - \frac{\partial\bar{\alpha}_{n-1}}{\partial z_{[n-1]}}\breve{f}_{[n-1]}(z_{[n-1]})\\ \bar{g}_n(z) &:= g_n(\Phi^{-1}(z)) - \frac{\partial\bar{\alpha}_{n-1}}{\partial z_{[n-1]}}\bar{G}_{1[n-1]}(z_{[n-1]}).\end{aligned}$$

$\breve{\Psi}(z) := \Phi^{-1}(z) - L^{-1}z$ contains only higher-order terms. With this procedure we have completed the construction of the diffeomorphism $z = \Phi(x)$ and transformed the original system (1) into the form

$$\dot{z} = \bar{A}z + \breve{f}(z) + \bar{G}_1(z)w + B_2 u.\tag{19}$$

The linear part of this diffeomorphism is Lx, that is $z = Lx + \breve{\Phi}(x)$, where $\breve{\Phi}(x) := \Phi(x) - Lx$ contains only higher-order terms. Thus, in both z and x coordinates, the properties of the linearized systems are the same.

D. Inverse Optimal Design

We now complete the design by constructing a nonlinear control law which attenuates the effects of the harmful nonlinearities and disturbance, and achieves global inverse optimality. The properties of the constructed control law are summarized in the following theorem.

Theorem 1: There exists a positive definite function $\bar{q}(z)$ and a strictly positive function $\bar{r}(z)$ with properties $\bar{q}_{zz}(0) = \bar{Q} = \bar{Q}' > 0$ and $\bar{r}(0) = R > 0$, such that the feedback control law

$$u = \bar{\mu}(z) := -\bar{r}^{-1}(z)B_2'\Delta z, \tag{20}$$

applied to the system (19) achieves local optimality with respect to the cost functional (15) and global inverse optimality with respect to a cost functional of the form

$$J(u,w) = \int_0^\infty \left[\bar{q}(z) + \bar{r}(z)u^2 - \gamma^2 w'w\right] dt, \tag{21}$$

for the worst-case disturbance $w = \bar{\nu}(z) := \frac{1}{\gamma^2}\bar{G}_1'(z)\Delta z$. Furthermore, $z(t) \to 0$ as $t \to \infty$ for all $w(t) \in \mathcal{L}_2$, and $z(t) \in \mathcal{L}_\infty$ for all $w(t) \in \mathcal{L}_\infty$. In the absence of a disturbance, $w \equiv 0$, the control law (20) renders the equilibrium $z = 0$ globally asymptotically stable (GAS) and locally exponentially stable (LES).

Proof: The CLF $\bar{V} = z'\Delta z > 0$ for system (19), constructed through the backstepping procedure of Section IV-C, has the derivative given by the formula (18) with $i = n$ and $x_{n+1} := u$, $\bar{\alpha}_n := 0$, namely

$$\begin{aligned}\dot{\bar{V}} &= 2z'\Delta\bar{A}z + \gamma^2\bar{\nu}_{l,n-1}'\bar{\nu}_{l,n-1} - \gamma^2\bar{\nu}_{n-1}'\bar{\nu}_{n-1}\\ &\quad +2\gamma^2\bar{\nu}'w + 2z_n\delta_n\left(\check{f}_n + u\right),\end{aligned}$$

where $\bar{\nu}(z) := \bar{\nu}_{n-1} + \frac{1}{\gamma^2}\bar{g}_n'\delta_n z_n = \frac{1}{\gamma^2}\bar{G}_1'(z)\Delta z$. Using the GARE identity (13), we reduce $\dot{\bar{V}}$ to

$$\begin{aligned}\dot{\bar{V}} &= -z'\bar{Q}z + R^{-1}\delta_n^2 z_n^2\\ &\quad -\gamma^2\bar{\nu}_{n-1}'\bar{\nu}_{n-1} - \gamma^2\left(\bar{\nu}_l'\bar{\nu}_l - \bar{\nu}_{l,n-1}'\bar{\nu}_{l,n-1}\right)\\ &\quad +2\gamma^2\bar{\nu}'w + 2z_n\delta_n\left(\check{f}_n + u\right).\end{aligned}$$

Adding $\bar{r}(z)u^2 - \bar{r}(z)u^2 + \gamma^2 w'w - \gamma^2 w'w \equiv 0$, and completing the squares with respect to both u and w we obtain

$$\begin{aligned}\dot{\bar{V}} &= -z'\bar{Q}z - \bar{r}(z)u^2 + \gamma^2 w'w - \gamma^2\left|w - \bar{\nu}\right|^2\\ &\quad +\bar{r}(z)\left(u - \bar{\mu}\right)^2 - \left(\bar{r}^{-1}(z) - R^{-1}\right)\delta_n^2 z_n^2\\ &\quad +\gamma^2\left(\bar{\nu}'\bar{\nu} - \bar{\nu}_{n-1}'\bar{\nu}_{n-1}\right) - \gamma^2\left(\bar{\nu}_l'\bar{\nu}_l - \bar{\nu}_{l,n-1}'\bar{\nu}_{l,n-1}\right)\\ &\quad +2z_n\delta_n\check{f}_n,\\ &= -z'\bar{Q}z - \bar{r}(z)u^2 + \gamma^2 w'w - \gamma^2\left|w - \bar{\nu}\right|^2\\ &\quad +\bar{r}(z)\left(u - \bar{\mu}\right)^2 - \left(\bar{r}^{-1}(z) - R^{-1}\right)\delta_n^2 z_n^2 + 2z_n\delta_n\check{\eta},\end{aligned}$$

where $\bar{\mu}(z)$ is defined by (20), and $\check{\eta}(z)$ contains only higher-order terms:

$$\begin{aligned}\check{\eta}(z) &:= \check{f}_n + \left(\bar{g}_n\bar{\nu}_{n-1} - \bar{b}_n\bar{\nu}_{l,n-1}\right)\\ &\quad +\frac{1}{2\gamma^2}\left(\bar{g}_n\bar{g}_n' - \bar{b}_n\bar{b}_n'\right)\delta_n z_n.\end{aligned}$$

The functions $\bar{r}(z)$ and $\bar{q}(z)$ are to be selected to satisfy the dissipation inequality

$$\dot{\bar{V}} \le -\bar{q}(z) - \bar{r}(z)u^2 + \gamma^2 w'w, \tag{22}$$

and to achieve global inverse optimality with respect to cost functional (21). For a meaningful cost $J(u,w)$, we restrict $\bar{q}(z)$ to be positive definite and $\bar{r}(z)$ to be positive for all z. Furthermore, we achieve local optimality by letting $\bar{r}(0) = R > 0$ and $\bar{q}_{zz}(0) = \bar{Q} = \bar{Q}' > 0$. One such choice of $\bar{q}(z)$ is

$$\bar{q}(z) := z'\bar{Q}z + \left(\bar{r}^{-1}(z) - R^{-1}\right)\delta_n^2 z_n^2 - 2z_n\delta_n\check{\eta}, \tag{23}$$

where $\bar{r}(z) > 0$ is to be constructed such that $\bar{q}(z)$ is positive definite and $\bar{r}(0) = R$.

We begin to construct $\bar{r}(z)$ by factoring out the terms of $\check{\eta}(z)$ and $z'\bar{Q}z$ that depend on z_n:

$$\begin{aligned}\check{\eta}(z) &= \bar{\eta}_1(z_{[n-1]})z_{[n-1]} + \bar{\eta}_2(z)z_n\\ z'\bar{Q}z &= \begin{bmatrix} z_{[n-1]}' & z_n\end{bmatrix}\begin{bmatrix}\bar{Q}_{[n-1]} & \bar{q}_1\\ \bar{q}_1' & \bar{q}_2\end{bmatrix}\begin{bmatrix} z_{[n-1]}\\ z_n\end{bmatrix},\end{aligned}$$

where $\bar{\eta}_1(0) = 0$ and $\bar{\eta}_2(0) = 0$. Next, we complete the squares in (23) with respect to $z_{[n-1]}$ and select $\bar{r}(z)$ to dominate only those terms in $\bar{q}(z)$ which are negative and which have not already been dominated by other positive terms:

$$\begin{aligned}\bar{q}(z) &= \left|z_{[n-1]} + \bar{Q}_{[n-1]}^{-1}(\bar{q}_1 - \bar{\eta}_1'\delta_n)z_n\right|^2_{\bar{Q}_{[n-1]}}\\ &\quad + \left(\bar{q}_2 - \bar{q}_1'\bar{Q}_{[n-1]}^{-1}\bar{q}_1\right)z_n^2\\ &\quad +(\bar{r}^{-1}(z) - R^{-1} - \bar{\sigma}(z))\delta_n^2 z_n^2,\end{aligned}$$

where

$$\bar{\sigma}(z) := \bar{\eta}_1\bar{Q}_{[n-1]}^{-1}\bar{\eta}_1' - 2\bar{q}_1'\bar{Q}_{[n-1]}^{-1}\delta_n^{-1}\bar{\eta}_1' + 2\delta_n^{-1}\bar{\eta}_2. \tag{24}$$

Clearly $\bar{\sigma}(0) = 0$. If $\bar{r}^{-1}(z) \ge R^{-1} + \bar{\sigma}(z)$, then $\bar{q}(z)$ is guaranteed to be positive for all $z \neq 0$, in fact,

$$\bar{q}(z) \ge \left\{\begin{array}{lcl}\lambda_{\min}\left(\bar{Q}_{[n-1]}\right)\left|z_{[n-1]}\right|^2 & : & z_n = 0\\ \left(\bar{q}_2 - \bar{q}_1'\bar{Q}_{[n-1]}^{-1}\bar{q}_1\right)\left|z_n\right|^2 & : & z_n \neq 0\end{array}\right\} > 0,$$

for all $z \neq 0$. Moreover, if $\bar{r}(0) = R$ then $\bar{q}_{zz}(0) = \bar{Q}$. Therefore, any positive function $\bar{r}(z) > 0$ such that $\bar{r}(0) = R$ and $\bar{r}^{-1}(z) \ge R^{-1} + \bar{\sigma}(z)$ will achieve our dual objective of local optimality and global inverse optimality.

Although the state penalty function $\bar{q}(z)$ is guaranteed to be positive definite, it may not be radially unbounded. If we require $\bar{q}(z)$ to be radially unbounded, then we must insure that $\bar{r}^{-1}(z) \ge R^{-1} + \bar{\sigma}(z) + \bar{\sigma}_r(z_{[n-1]}) \ge R^{-1} + \bar{\sigma}(z)$, where $\bar{\sigma}_r(z_{[n-1]})$ is any positive definite and radially unbounded function such that $\bar{\sigma}_r(0) = 0$. If this is the case, then we simply absorb the function $\bar{\sigma}_r(z_{[n-1]})$ into $\bar{\sigma}(z)$,

$$\bar{\sigma}(z) + \bar{\sigma}_r(z_{[n-1]}) \to \bar{\sigma}(z),$$

and continue with the design process. One particular choice, $\bar{\sigma}_r(z_{[n-1]}) = \left|z_{[n-1]}\right|^2$, guarantees the existence of a positive constant c_q such that $\bar{q}(z) \geq c_q \bar{V}(z)$ for all $z \in \mathbb{R}^n$.

We identify two particular choices of $\bar{r}(z)$. The first choice is

$$\bar{r}(z) := \left(\sqrt{R^{-2} + \bar{\sigma}^2(z)} + \bar{\sigma}(z)\right)^{-1}.$$

It is clear that $\bar{r}(z)$ is a smooth and positive function, $\bar{r}(0) = R$, and $\bar{r}^{-1}(z) \geq R^{-1} + \bar{\sigma}(z)$, $\forall z \in \mathbb{R}^n$. The second choice,

$$\bar{r}(z) := \begin{cases} \left(R^{-1} + \bar{\sigma}(z)\right)^{-1} & : \quad \bar{\sigma}(z) \geq 0 \\ R & : \quad -R^{-1} \leq \bar{\sigma}(z) < 0 \\ \bar{\epsilon}(\bar{\sigma})R & : \quad \bar{\sigma}(z) < -R^{-1}, \end{cases} \tag{25}$$

requires $\bar{r}^{-1}(z)$ to dominate $R^{-1} + \bar{\sigma}(z)$ only when necessary, and is therefore *leaner* than the first. However, it may not be differentiable everywhere. The function $\bar{\epsilon}(\bar{\sigma})$ can be any continuous and strictly positive function such that $\bar{\epsilon}(-R^{-1}) = 1$. One such function is $\bar{\epsilon}(\bar{\sigma}) := 1 + \left|R^{-1} + \bar{\sigma}(z)\right|^k$ for any $k > 0$. For either of the two choices,

$$\begin{aligned} \dot{\bar{V}} &= -\bar{q}(z) - \bar{r}(z)u^2 + \gamma^2 w'w \\ &\quad -\gamma^2 \left|w - \bar{\nu}\right|^2 + \bar{r}(z)\left(u - \bar{\mu}\right)^2. \end{aligned} \tag{26}$$

It is obvious from (26) that the optimal control $u = \bar{\mu}(z)$ and the worst-case disturbance $w = \bar{\nu}(z)$ satisfy

$$\min_u \max_w \left[\bar{q}(z) + \bar{r}(z)u^2 - \gamma^2 w'w + \dot{\bar{V}}(z)\right] = 0,$$

and that $\bar{V}(z)$ satisfies the dissipation inequality (22) and the HJI equation

$$\begin{aligned} 0 &= \bar{q}(z) + \bar{V}_z(z)\left(\bar{A}z + \check{f}(z)\right) \\ &+ \frac{1}{4}\bar{V}_z(z)\left(\frac{1}{\gamma^2}\bar{G}_1(z)\bar{G}_1'(z) - B_2\bar{r}^{-1}(z)B_2'\right)\bar{V}_z'(z). \end{aligned}$$

Therefore, $\bar{V}(z)$ is the value function for the cost functional (21). Likewise, $V(x) =: \bar{V}(\Phi(x))$ is the value function in the x-coordinates.

If $\bar{q}(z)$ is radially unbounded, then the system is input-to-state stable [10] from w to z. Hence, the system state remains bounded for all bounded disturbances. Furthermore, since $\dot{\bar{V}} \leq -\bar{q}(z) + \gamma^2 w'w$ and $\bar{q}(z)$ is zero only if $z = 0$, we have $z(t) \in \mathcal{L}_\infty$ and $z(t) \to 0$ ($x(t) \to 0$) as $t \to \infty$ whenever $w(t) \in \mathcal{L}_2$ [19]. A summary of the design procedure is found in Table I

V. Example

We illustrate the steps of the design procedure on the simple nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 + w \\ \dot{x}_2 &= u, \end{aligned} \tag{27}$$

and cost functional

$$J = \int_0^\infty \left[x_1^2 + x_2^2 + u^2 - 5^2 w^2\right] dt.$$

Let us first examine the stability properties of the $\mathcal{H}_\infty$-optimal controller for the linearized system

$$\begin{aligned} \dot{x}_1 &= x_2 + w \\ \dot{x}_2 &= u. \end{aligned} \tag{28}$$

The optimal attenuation level for this problem is $\gamma^* = 1.27$. Since $\gamma = 5 > \gamma^*$, the GARE (9) has a unique positive definite solution $P = \begin{bmatrix} 1.82 & 1.06 \\ 1.06 & 1.78 \end{bmatrix} > 0$, and the optimal controller (10) for (28) is

$$u = -B'Px = -1.06x_1 - 1.78x_2. \tag{29}$$

When controller (29) is applied to the nonlinear system (27), the resulting feedback system is

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2 + w \\ \dot{x}_2 &= -1.06x_1 - 1.78x_2. \end{aligned}$$

The disturbance-free phase-portrait of this system is represented by the dashed trajectories in Figure 1. The key instability property is the presence of an unstable equilibrium (saddle-point) at $x_1 = 0.60$, $x_2 = -0.35$. As shown in Figure 1, the stability boundary of the desired equilibrium $x_1 = x_2 = 0$ is the stable manifold of the saddle-point equilibrium denoted by M_s. The region of attraction of $x = 0$ is to the left of this manifold. All trajectories to the right of M_s are unbounded. For initial conditions in the region of attraction which are close to the manifold M_s, the transient performance is poor. Under the action of the worst-case disturbance, the region of attraction would be much smaller.

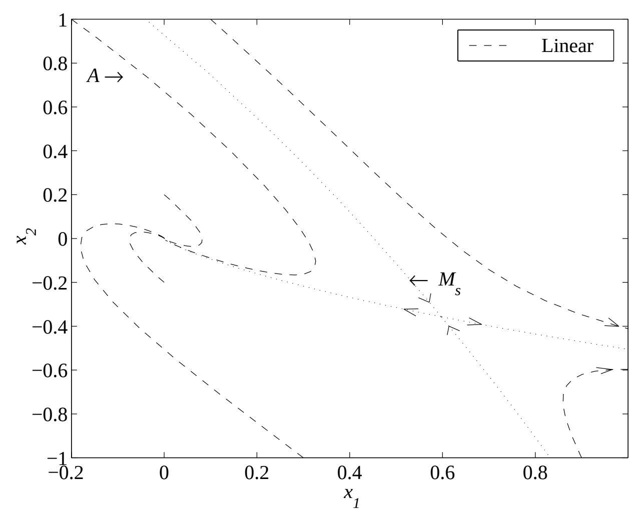

Fig. 1. Linear Phase-Plane Trajectories.

When a linear optimal design fails as in the above example, the design presented in this paper provides a systematic way to improve both the performance and stability.

We now proceed with the new design in which we first recover (29) as the linear part of the nonlinear controller. As shown in Section IV-A the key to achieving local optimality in backstepping designs is the factorization $P = L'\Delta L$ of the positive definite solution of the GARE (9). The matrices L and Δ uniquely define the constants α_{11}, δ_1 and δ_2 which are needed in the locally optimal backstepping design:

$$L = \begin{bmatrix} 1 & 0 \\ -\alpha_{11} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.6 & 1 \end{bmatrix}$$

and

$$\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} = \begin{bmatrix} 1.18 & 0 \\ 0 & 1.78 \end{bmatrix}.$$

Under the transformation $z = Lx$, that is $z_1 = x_1$, $z_2 = x_2 + 0.6x_1$, the optimal control law (17) for the linear system is $u = -B'\Delta z = -1.78z_2$, or equivalently, $u = -1.78\,(0.6x_1 + x_2) = -1.06x_1 - 1.78x_2$, which is the same as (29).

Carrying out the first step of the locally optimal backstepping procedure of Section IV-C, we define $z_1 = \phi_1(x_1) := x_1$. Then, as dictated by the linear design and the factorization of P, we select $\bar{V}_1 = 1.18z_1^2$ as the value function, and $\bar{\alpha}_1(z_1) := -0.6z_1 + \breve{\alpha}_1(z_1)$ as the virtual control law for x_2. Adding $25w^2 - 25w^2 \equiv 0$ to the derivative of $\bar{V}_1$, and completing the squares with respect to w, we compute

$$\begin{aligned} \dot{\bar{V}}_1 &= -1.36z_1^2 + 25w^2 - 25\,(w - \bar{\nu}_1)^2 \\ &\quad +2.36z_1\left[z_1^2 + \breve{\alpha}_1\right] + 2.36z_1\,(x_2 - \bar{\alpha}_1)\,, \end{aligned}$$

where $\bar{\nu}_1(z_1) := 0.05z_1$. The nonlinear virtual control term $\breve{\alpha}_1(z_1)$ is selected to cancel the nonlinearity in brackets, that is, $\breve{\alpha}_1(z_1) = -z_1^2$. Now,

$$\begin{aligned} \dot{\bar{V}}_1 &= -1.36z_1^2 + 25w^2 - 25\,(w - \bar{\nu}_1)^2 \\ &\quad +2.36z_1\,(x_2 - \bar{\alpha}_1)\,, \end{aligned}$$

and when $x_2 \equiv \bar{\alpha}_1(z_1)$, $\bar{V}_1$ satisfies the dissipation inequality

$$\dot{\bar{V}}_1 \leq -1.36z_1^2 + 25w^2.$$

At the second step, we define $z_2 = \phi_2(x) := x_2 - \bar{\alpha}_1(z_1)$ and select $\bar{V} = \bar{V}_1 + 1.78z_2^2$ as the value function. With this final step we have constructed the diffeomorphism

$$z := \begin{bmatrix} x_1 \\ 0.6x_1 + x_2 + x_1^2 \end{bmatrix} = \Phi(x) = Lx + \begin{bmatrix} 0 \\ x_1^2 \end{bmatrix}. \tag{30}$$

Because the linear part of the (30) is $z = Lx$, the local properties of the linear design are preserved. Using (30), we obtain the nonlinear system dynamics in the z-coordinates:

$$\begin{aligned} \dot{z}_1 &= -0.6z_1 + z_2 + w \\ \dot{z}_2 &= -0.36z_1 + 0.6z_2 - 1.2z_1^2 + 2z_1z_2 + u \\ &\quad + (2z_1 + 0.6)\,w. \end{aligned}$$

The derivative of the value function $\bar{V}$ is

$$\dot{\bar{V}} = 2z'\Delta\bar{A}z + 50\bar{\nu}w + 3.56\left(-1.19z_1^2 + 2z_1z_2 + u\right)z_2,$$

where $\bar{A} = LAL^{-1} = \begin{bmatrix} -0.6 & 1 \\ -0.36 & 0.6 \end{bmatrix}$ and the worst-case disturbance is

$$\begin{aligned} \bar{\nu}(z) &:= \bar{\nu}_1 + 0.07\,(2z_1 + 0.6)\,z_2 \\ &= 0.05z_1 + 0.04z_2 + 0.14z_1z_2. \end{aligned}$$

Using the GARE identity (13) with $\bar{Q} = L'^{-1}QL^{-1} = \begin{bmatrix} 1.36 & -0.6 \\ -0.6 & 1 \end{bmatrix}$, and completing the squares with respect to both w and u, we obtain

$$\begin{aligned} \dot{\bar{V}} &= -\bar{q}(z) - \bar{r}(z)u^2 + 25w^2 \\ &\quad -25\,(w - \bar{\nu})^2 + \bar{r}(z)\,(u - \bar{\mu})^2\,, \end{aligned}$$

where

$$\bar{\mu}(z) := -1.78\bar{r}^{-1}(z)z_2, \tag{31}$$

and

$$\begin{aligned} \bar{q}(z) &:= z'\bar{Q}z + 3.17\left(\bar{r}^{-1}(z) - 1\right)z_2^2 \\ &\quad -3.56\bar{\eta}_1(z_1)z_1z_2 - 3.56\bar{\eta}_2(z)z_2^2 \\ \bar{\eta}_1(z_1) &:= -1.10z_1 \\ \bar{\eta}_2(z) &:= 2.08z_1 + 0.09z_1^2. \end{aligned}$$

After completing the squares in $\bar{q}(z)$ with respect to z_1, we define $\bar{\sigma}(z)$ as prescribed by (24): $\bar{\sigma}(z) = 1.8z_1 + 1.05z_1^2$. We now rewrite $\bar{q}(z)$ as

$$\begin{aligned} \bar{q}(z) &= 1.36\,(z_1 - 0.44z_2 + 1.44z_1z_2)^2 + 0.74z_2^2 \\ &\quad +3.17(\bar{r}^{-1}(z) - 1 - \bar{\sigma}(z))z_2^2. \end{aligned}$$

If $\bar{q}(z)$ needs to be radially unbounded, or $w(t) \in \mathcal{L}_\infty$, then we let $\bar{\sigma}(z) + \bar{\sigma}_r(z_1) \to \bar{\sigma}(z)$ where $\bar{\sigma}_r(z) := z_1^2$, and continue with the design process.

The feedback law (31) will stabilize the system with either of the two choices for $\bar{r}(z)$ introduced in Section IV-D. We select $\bar{r}(z)$ as defined by (25) with $\bar{\epsilon}(\bar{\sigma}) = 1$ and the feedback control law (31) becomes

$$u = \begin{cases} -1.78\,(1 + \bar{\sigma}(z))\,z_2 & : \ \bar{\sigma}(z) \geq 0 \\ -1.78z_2 & : \ \bar{\sigma}(z) < 0. \end{cases} \tag{32}$$

Since $\sigma(x) := \bar{\sigma}(\Phi(x))$, we can write (32) in the x-coordinates (7) as

$$u = \begin{cases} -1.78\,(1 + \sigma(x))\left(0.6x_1 + x_2 + x_1^2\right) & : \sigma(x) \geq 0 \\ -1.78\left(0.6x_1 + x_2 + x_1^2\right) & : \sigma(x) < 0. \end{cases} \tag{33}$$

Solid trajectories in Figure 2 represent the phase-portrait of the disturbance-free system (27) controlled by the nonlinear controller (33). Unlike the linear controller, trajectories of the nonlinear controller remain bounded for all initial conditions. The nonlinear controller achieves not

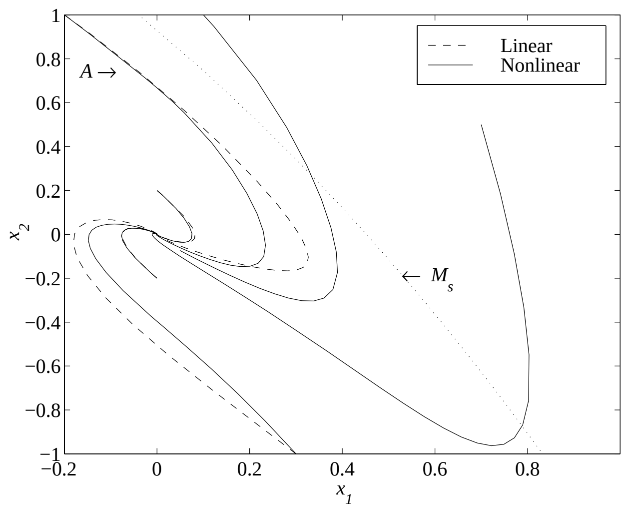

Fig. 2. Phase-Plane Trajectories.

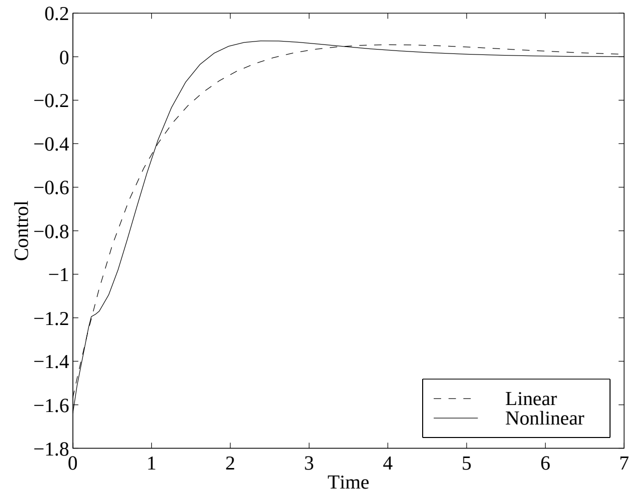

Fig. 3. Control history for trajectory A.

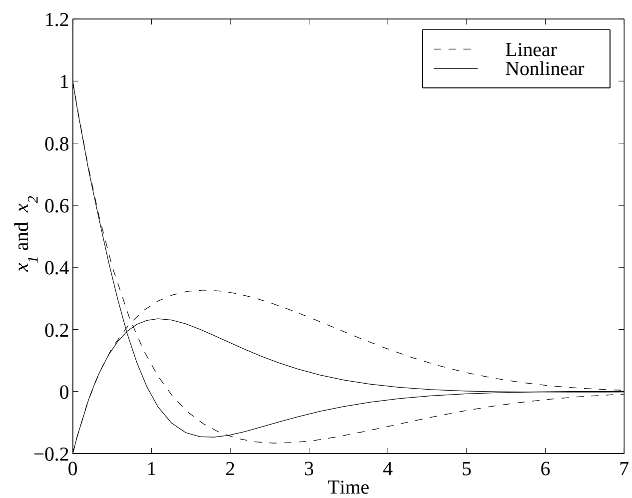

Fig. 4. State history for trajectory A.

only global asymptotic stability, but it also improves the performance without an increase in control effort. This is illustrated in Figure 3 for trajectory A which is close to the stability boundary M_s. The control effort required by both the linear and nonlinear controllers is comparable. As shown in Figure 4, the states of the system driven by the nonlinear controller converge to zero much faster than those of the system driven by the linear controller. This difference is more pronounced when the initial condition is closer to M_s. The local optimality is illustrated in Figure 2 by the fact that the trajectories of both controllers are almost identical for initial conditions close to the origin.

In the presence of a bounded disturbance, the nonlinear controller (33) guarantees boundedness, and the $\mathcal{L}_2$-gain from the disturbance w to the controlled output $\mathfrak{z} = \left(q(x) + r(x)u^2\right)^{1/2}$ is bounded from above by $\gamma = 5$. Thus, since $q_{xx}(0) = I$ and $r(0) = 1$, the nonlinear controller has preserved the good local properties of the linear controller while attaining robust stability.

VI. Conclusions

The new backstepping procedure presented in this paper simultaneously achieves several design objectives: local optimality with respect to a prescribed quadratic cost, global inverse optimality with respect to a cost functional constructed during the design, and global $\mathcal{L}_2$-disturbance attenuation. In the presence of a bounded disturbance, the system state remains bounded if the weighting function $q(x)$ is designed to be radially unbounded, and when $w(t) \in \mathcal{L}_2$, all trajectories converge to zero. In the absence of a disturbance, global asymptotic stability of the equilibrium is guaranteed. In this way, the design freedoms of earlier robust backstepping procedures [4], [8], [9] are now exploited to systematically improve both performance and stability properties of nonlinear feedback systems.

The locally optimal backstepping design has been extended in [14] using a "nonlinear Cholesky factorization" which permits the design of control laws that are globally inverse $\mathcal{H}_\infty$-optimal and locally $\mathcal{H}_\infty$-optimal with respect to *non-quadratic* cost functionals. The locally optimal design has also been applied to systems which require *vectorial* backstepping such as ships [20]. The problem of obtaining local optimality when the linear system is uncontrollable remains to be solved.

Finally, it should be noted that the control laws generated by the procedure presented in this paper may exhibit gains in certain regions of the state-space which are unnecessarily high. This "hardening" phenomenon is related to the quadratic choice of the control Lyapunov function, and can be reduced in standard backstepping through the use of "flattened" Lyapunov functions [8]. Therefore, as long as the control Lyapunov function remains quadratic in a ball containing $x = 0$, it should be possible to modify the "flattened" Lyapunov function approach to accommodate the locally optimal backstepping design, and thus reduce the affects of hardening.

TABLE I

THE DESIGN PROCEDURE, $i = 1, \ldots, n$.

Diffeomorphism: Section IV-C, $z = \Phi(x)$

$$z_i = \phi_i(x_{[i]}) = x_i - \bar{\alpha}_{i-1}(z_{[i-1]}) = x_i - \bar{\alpha}_{[i-1]} z_{[i-1]} - \check{\alpha}_{i-1}(z_{[i-1]})$$

Linear Factorization: Section IV-A, $\bar{\nu}_{l0} := 0$

$$0 = PA + A'P + P\left(\frac{1}{\gamma^2}B_1B_1' - B_2R^{-1}B_2'\right)P + Q \quad \Rightarrow \quad P \equiv L'\Delta L > 0$$

$$\begin{aligned} \bar{A} &= LAL^{-1}, \quad \bar{B}_1 = LB_1, \quad \bar{Q} = L'^{-1}QL^{-1} \\ \bar{\nu}_{li} &= \frac{1}{\gamma^2}\bar{B}'_{1[i]}\Delta_{[i]}z_{[i]} \end{aligned}$$

L and Δ determine the value of α_{ij} and δ_i for $i, j = 1, \ldots, n$

Local Design: Section IV-B, $\alpha_{[0]} := \emptyset$, $\alpha_{[n]} := \emptyset$

$$\begin{aligned} \bar{\alpha}_{[i]} &= \alpha_{[i]}L_{[i]}^{-1} \\ \bar{a}_{[i]} &\equiv a_{[i]}L_{[i]}^{-1} + \bar{\alpha}_{[i]} - \begin{bmatrix}\bar{\alpha}_{[i-1]}\bar{A}_{[i-1]} & \bar{\alpha}_{i-1,i-1}\end{bmatrix} \\ \bar{b}_i &\equiv b_i - \bar{\alpha}_{[i-1]}\bar{B}_{1[i-1]} \end{aligned}$$

Global Design: Section IV-C, $\check{\alpha}_0 := 0$, $\check{\alpha}_n := 0$, $\bar{\nu}_0 := 0$

$$\begin{aligned} \check{\alpha}_i &= -a_{[i]}\check{\Psi}_{[i]}(z_{[i]}) - \check{f}_i(\Phi_{[i]}^{-1}(z_{[i]})) + \frac{\partial\check{\alpha}_{i-1}}{\partial z_{[i-1]}}\left[\bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix}0_{i-2}\\ z_i\end{bmatrix}\right] \\ &\quad + \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\check{f}_{[i-1]}(z_{[i-1]}) - \left(\bar{g}_i\bar{\nu}_{i-1} - \bar{b}_i\bar{\nu}_{l,i-1}\right) - \frac{1}{2\gamma^2}\left(\bar{g}_i\bar{g}_i' - \bar{b}_i\bar{b}_i'\right)\delta_i z_i \\ \check{f}_i &= \check{\alpha}_i(z_{[i]}) + a_{[i]}\check{\Psi}_{[i]}(z_{[i]}) + \check{f}_i(\Phi_{[i]}^{-1}(z_{[i]})) - \frac{\partial\check{\alpha}_{i-1}}{\partial z_{[i-1]}}\left[\bar{A}_{[i-1]}z_{[i-1]} + \begin{bmatrix}0_{i-2}\\ z_i\end{bmatrix}\right] \\ &\quad - \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\check{f}_{[i-1]}(z_{[i-1]}) \\ \bar{g}_i &= g_i(\Phi_{[i]}^{-1}(z_{[i]})) - \frac{\partial\bar{\alpha}_{i-1}}{\partial z_{[i-1]}}\bar{G}_{1[i-1]}(z_{[i-1]}) \\ \bar{\nu}_i &= \bar{\nu}_{i-1} + \frac{1}{\gamma^2}\bar{g}_i'\delta_i z_i \\ \check{\Psi}_{[i]} &= \Phi_{[i]}^{-1}(z_{[i]}) - L_{[i]}^{-1}z_{[i]} \end{aligned}$$

Inverse Optimal Design: Section IV-D, $J(u,w) = \int_0^\infty \left[\bar{q}(z) + \bar{r}(z)u^2 - \gamma^2 w'w\right]dt$

$$\begin{aligned} \bar{r} &= \left(\sqrt{R^{-2} + \bar{\sigma}^2(z)} + \bar{\sigma}(z)\right)^{-1} > 0 \\ \bar{q} &= \left|z_{[n-1]} + \bar{Q}_{[n-1]}^{-1}(\bar{q}_1 - \bar{\eta}_1'\delta_n)z_n\right|^2_{\bar{Q}_{[n-1]}} + \left(\bar{q}_2 - \bar{q}_1'\bar{Q}_{[n-1]}^{-1}\bar{q}_1\right)z_n^2 + (\bar{r}^{-1} - R^{-1} - \bar{\sigma})\delta_n^2 z_n^2 \geq 0 \\ \bar{\sigma} &= \bar{\eta}_1\bar{Q}_{[n-1]}^{-1}\bar{\eta}_1' - 2\bar{q}_1'\bar{Q}_{[n-1]}^{-1}\delta_n^{-1}\bar{\eta}_1' + 2\delta_n^{-1}\bar{\eta}_2 + \bar{\sigma}_r(z_{[n-1]}) \\ \check{\eta} &= \check{f}_n + \left(\bar{g}_n\bar{\nu}_{n-1} - \bar{b}_n\bar{\nu}_{l,n-1}\right) + \frac{1}{2\gamma^2}\left(\bar{g}_n\bar{g}_n' - \bar{b}_n\bar{b}_n'\right)\delta_n z_n \equiv \bar{\eta}_1(z_{[n-1]})z_{[n-1]} + \bar{\eta}_2(z)z_n \end{aligned}$$

$\bar{\sigma}_r(z_{[n-1]})$ is any positive definite and radially unbounded function such that $\bar{\sigma}_r(0) = 0$

Control Law: Section IV-D, $\bar{V} = z'\Delta z \geq 0$

$$u = -\bar{r}^{-1}(z)\delta_n z_n = -\bar{r}^{-1}(z)B_2'\Delta z$$

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Acknowledgments

The authors would like to express their gratitude to Professor Arthur J. Krener for stimulating discussions regarding local optimality.

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