Disturbance Attenuating Output-Feedback Control of Nonlinear Systems with Local Optimality [1]

Kenan Ezal
Toyon Research Corporation
75 Aero Camino, Suite A
Goleta, CA 93117
kezal@toyon.com

Petar V. Kokotović, Andrew R. Teel
Center for Control Engineering and Computation
University of California
Santa Barbara, CA 93106
petar@ece.ucsb.edu,
teel@iracema.ece.ucsb.edu

Tamer Başar
Coordinated Science Laboratory
University of Illinois
Urbana, IL 61801
tbasar@isaacs.csl.uiuc.edu

Abstract

Recent results on locally optimal and globally inverse optimal full-state feedback designs for strict-feedback systems are now extended for *output-feedback systems* in *output-feedback form*. The dynamic output-feedback control law for such systems is constructed under the assumption that the derivative of the measured output is available for feedback, and achieves local optimality and global inverse optimality with a prescribed $\mathcal{L}_2$-gain. When that assumption is subsequently dropped, the control law achieves semiglobal inverse optimality and local near-optimality.

1 Introduction

While $\mathcal{H}_\infty$-design of linear systems has been successful, its nonlinear counterpart has proven to be much more difficult. Nonlinear $\mathcal{H}_\infty$ results are either local in nature [7], yield infinite dimensional controllers [2, 1], or provide optimality conditions without any guidance as how to explicitly satisfy them [13]. Constructive nonlinear $\mathcal{H}_\infty$-designs, either do not achieve local $\mathcal{H}_\infty$-optimality, or do not penalize the cost of control [6, 12]. The main obstacle in achieving optimal control of nonlinear systems is the need to solve the Hamilton-Jacobi-Isaacs (HJI) equation. Because of this difficulty, recent research has concentrated on *inverse optimal* designs [6, 5].

In this paper, we present finite dimensional dynamic output-feedback controllers which achieve local (near) $\mathcal{H}_\infty$-optimality and (semi) global inverse $\mathcal{H}_\infty$-optimality with a prescribed $\mathcal{L}_2$-gain for the class of *strict-feedback systems* [9] in *output-feedback form.* While we employ the locally optimal backstepping design procedure of [5] to construct the output-feedback control law, we rely on nonlinear filter designs based on the *cost-to-come* methods introduced in [3] and [10, 12] to obtain a robust estimate of the unmeasured states.

2 Notation

Our notation is similar to that of [5], i.e., given $A \in \mathbb{R}^{n\times n}$, the i-th linear subsystem matrix is denoted as $A_{[i]}$. The same notation is used for the subsets of states $x_{[i]}$ and vector fields, i.e., $f_{[i]}$ and $G_{1[i]}$. We also define $A_{\{i+1\}}$, $A_{12\{i+1\}}$ and $A_{21\{i+1\}}$ such that

$$A \equiv \begin{bmatrix} A_{[i]} & A_{12\{i+1\}} \\ A_{21\{i+1\}} & A_{\{i+1\}} \end{bmatrix}.$$

[1] This research was supported in part by NSF under Grants ECS-9812346 and ECS-9896140, AFOSR under Grants F49620-95-1-0409 and F49620-98-1-0087, and the U.S. DOE under Grant DE-FG-02-97-ER-13939.

The sets $\mathcal{K}_\infty$, $\mathcal{L}_2$ and $\mathcal{L}_\infty$, and $O(\epsilon)$ are defined in the usual manner; 0_n is a zero column vector of length n; I_n is an $n\times n$ identity matrix; $\mathcal{P}_n$ is the set of symmetric positive definite $n \times n$ matrices; $\mathcal{P}_{\geq 0}(x)$ is the set of positive functions zero only at $x = 0$; and $\mathcal{P}_{>0}(x)$ is the set of strictly positive functions.

3 Problem Formulation

We consider systems in output-feedback form

$$\dot{x} = Ax + \breve{f}(x_1) + G_1(x_1)w + B_2u \quad (3.1a)$$
$$y = C_1x, \quad (3.1b)$$

where $x \in \mathbb{R}^n$ is the system state; $w \in \mathbb{R}^q$ is the unknown input disturbance; $u \in \mathbb{R}$ is the control input; $y \in \mathbb{R}$ is the measured output; A is a lower Hessenberg matrix with ones along the first upper diagonal; and $\breve{f}(x_1) = \begin{bmatrix}\breve{f}_1(x_1) & \cdots & \breve{f}_n(x_1)\end{bmatrix}'$, $G_1(x_1) = \begin{bmatrix}g_1'(x_1) & \cdots & g_n'(x_1)\end{bmatrix}'$, $B_1 := G_1(0)$, $b_i := g_i(0)$ for $i = 1,\ldots,n$, $B_2 = [0_{n-1}' \quad 1]' \in \mathbb{R}^n$, $C_1 = [1 \quad 0_{n-1}'] \in \mathbb{R}^{1\times n}$. We assume $\breve{f}(x_1)$ and $G_1(x_1)$ are sufficiently smooth, that $\breve{f}(x_1)$ does not contain any linear terms, and that $g_1(x_1)g_1'(x_1) > 0$ for all $x_1 \in \mathbb{R}$. Moreover, (A, B_2) is controllable and (A, C_1) is observable.

Given system (3.1), our task is to design an asymptotically stabilizing and disturbance attenuating dynamic output-feedback control law $\dot{\hat{x}} = \hat{F}(y, \hat{x}, u)$, $u = \mu(y, \hat{x})$, where $\hat{x} \in \mathbb{R}^n$ is the estimate of the state x, and $\tilde{x} = x - \hat{x}$ is the estimation error such that $\tilde{x}(t) \to 0$ as $t \to \infty$ for any $w(t) \in \mathcal{L}_2$. We consider two problems which differ in whether or not the derivative of the measurement output is available for feedback. In the first problem we achieve local $\mathcal{H}_\infty$-optimality with global inverse $\mathcal{H}_\infty$-optimality, while in the second case we achieve local near-optimality with semiglobal inverse $\mathcal{H}_\infty$-optimality.

3.1 Derivative Information Available

The cost-to-come methodology requires that the output y depend on the disturbance w in a non-singular way, a requirement not satisfied by system (3.1). In Section 4, we make the problem tractable with the following two assumptions.

Assumption 3.1 *The derivative of the output, $\dot{y}$, is available for feedback.*

Assumption 3.2 $N(x_1) := g_1(x_1)g_1'(x_1) > 0$ *for all* $x_1 \in \mathbb{R}$. *In the linear case,* $N_0 := b_1b_1' > 0$.

Our local objective is to achieve $\mathcal{H}_\infty$-optimality in the region of validity of the linear dynamics

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + x_2 + b_1 w_l && (3.2a)\\
\dot{x}_{\{2\}} &= A_{21\{2\}}x_1 + A_{\{2\}}x_{\{2\}}\\
&\quad + B_{1\{2\}}w_l + B_{2\{2\}}u_l && (3.2b)\\
y_l &= C_1 x && (3.2c)\\
\mathfrak{z}_l &= \hat{C}_{21}\tilde{x}_{\{2\}} + \hat{C}_{22}\underline{x} + D_2 u_l, && (3.2d)
\end{aligned}$$

where $\mathfrak{z}_l$ is the linear controlled output with $\underline{x} := \begin{bmatrix} x_1 & \hat{x}'_{\{2\}} \end{bmatrix}'$. Our first objective is to find a stabilizing control law $u_l = \mu_l(y_l, \hat{x}_{\{2\}})$ which achieves a prescribed $\mathcal{L}_2$-gain $\gamma > 0$ from w_l to $\mathfrak{z}_l$ under the following assumption.

Assumption 3.3 *$Q_1 := \hat{C}'_{21}\hat{C}_{21} > 0$, $Q_2 = \hat{C}'_{22}\hat{C}_{22} > 0$, $R = D'_2 D_2 > 0$, $\hat{C}'_{21}\hat{C}_{22} = 0$, $\hat{C}'_{21}D_2 = 0$ and $\hat{C}'_{22}D_2 = 0$, where $Q_1 \in \mathcal{P}_{n-1}$, $Q_2 \in \mathcal{P}_n$, and $R > 0$ are prespecified.*

This disturbance attenuation problem is equivalent to a dynamic game with

$$\hat{J}_l(u_l, w_l) = \int_0^\infty \left[\mathfrak{z}'_l\mathfrak{z}_l - \gamma^2 w'_l w_l\right] dt \qquad (3.3)$$

as the cost functional. We assume that the desired level of attenuation is achievable: $\gamma > \gamma^*$ where the finite optimal linear attenuation level $\gamma^* > 0$ is assumed to exist.

In linear $\mathcal{H}_\infty$-designs it is common to let $\mathfrak{z}_l = C_2 x + D_2 u$, where $Q := C'_2 C_2 > 0$ and $R = D'_2 D_2 > 0$ are prespecified and $C'_2 D_2 = 0$, that is

$$J_l(u_l, w_l) = \int_0^\infty \left[x'Qx + Ru_l^2 - \gamma^2 w'_l w_l\right] dt. \qquad (3.4)$$

We will instead achieve suboptimality with respect to (3.4). By suboptimality we mean that a control law which guarantees a certain level of disturbance attenuation with respect to a particular cost functional also guarantees the same level of disturbance attenuation with respect to any other cost functional that is no larger than the original cost functional. Hence, in order to achieve suboptimality with respect to (3.4) for any prespecified $Q \in \mathcal{P}_n$, we can always choose $Q_1 \in \mathcal{P}_{n-1}$ and $Q_2 \in \mathcal{P}_n$ such that

$$\tilde{x}'_{\{2\}}Q_1\tilde{x}_{\{2\}} + \underline{x}'Q_2\underline{x} \geq x'Qx. \qquad (3.5)$$

Like the standard $\mathcal{H}_\infty$-optimal output-feedback control problem, we assume the existence of stabilizing solutions to two generalized algebraic Riccati equations (GARE).

Assumption 3.4 *Given $Q_1 \in \mathcal{P}_{n-1}$, the generalized filtering algebraic Riccati equation*

$$\begin{aligned}
0 &= \Pi_s^\infty \hat{A}'_0 + \hat{A}_0\Pi_s^\infty + \gamma^{-2}\hat{B}_0\hat{B}'_0\\
&\quad -\Pi_s^\infty\left[\gamma^2\hat{C}'_1 N_0^{-1}\hat{C}_1 - Q_1\right]\Pi_s^\infty, \qquad (3.6)
\end{aligned}$$

with $\hat{C}_1 := [1 \quad 0'_{n-2}] \in \mathbb{R}^{1\times n-1}$, and

$$\begin{aligned}
\hat{A}_0 &:= A_{\{2\}} - B_{1\{2\}}b'_1 N_0^{-1}\hat{C}_1\\
\hat{B}_0 &:= \left(B_{1\{2\}}\left[I_q - b'_1 N_0^{-1} b_1\right]B'_{1\{2\}}\right)^{1/2},
\end{aligned}$$

admits a symmetric positive definite solution Π_s^∞ such that $\hat{A}_0 - \gamma^2\Pi_s^\infty\hat{C}'_1 N_0^{-1}\hat{C}_1$ is Hurwitz.

Assumption 3.5 *Given $Q_2 \in \mathcal{P}_n$ and $R > 0$, the generalized control algebraic Riccati equation*

$$\begin{aligned}
0 &= P_s A + A'P_s + Q_2\\
&\quad + P_s\left(\frac{1}{\gamma^2}\hat{B}_1\hat{B}'_1 - B_2 R^{-1}B'_2\right)P_s, \qquad (3.7)
\end{aligned}$$

with $\hat{B}_1 := \begin{bmatrix} N_0^{1/2} & \hat{\Gamma}'_{1_0}\end{bmatrix}'$, and

$$\hat{\Gamma}_{1_0} = \left(B_{1\{2\}}b'_1 + \gamma^2\Pi_s^\infty\hat{C}'_1\right)N_0^{-1/2}$$

admits a symmetric solution $P_s \in \mathcal{P}_n$ such that

$$A - \left(B_2 R^{-1}B'_2 - \frac{1}{\gamma^2}\hat{B}_1\hat{B}'_1\right)P_s$$

and $A - B_2 R^{-1}B'_2 P_s$ are Hurwitz.

Our global objective is to achieve inverse $\mathcal{H}_\infty$-optimality of the nonlinear system with respect to the cost functional of the form

$$\begin{aligned}
\hat{J}(u, w) &= \int_0^\infty \left[\mathfrak{z}'\mathfrak{z} - \gamma^2 w'w\right] dt\\
&\quad + \left|\tilde{x}_{\{2\}}(0)\right|^2_{(\Pi_s^\infty)^{-1}} + l_0(x_1(0)), \qquad (3.8)
\end{aligned}$$

where $\gamma > \gamma^*$ and $\mathfrak{z}$ is the controlled output such that

$$|\mathfrak{z}(t)|^2 = q_1(\tilde{x}_{\{2\}}, t) + q_2(\underline{x}, t) + r(\underline{x}, t)u^2 \geq 0, \qquad (3.9)$$

$q_1(\tilde{x}_{\{2\}}, t) := \tilde{x}'_{\{2\}}\Theta_s(t)\tilde{x}_{\{2\}} \geq 0$, $q_2(\underline{x}, t) \geq 0$, and $r(\underline{x}, t) > 0$. While $q_1(\tilde{x}_{\{2\}}, t)$ and $q_2(\underline{x}, t)$ are positive definite in $\tilde{x}_{\{2\}}$ and $\underline{x}$ respectively, $r(\underline{x}, t)$ is strictly positive. In order to simultaneously satisfy both the local optimality objective and the global inverse optimality objective, we impose the additional requirements

$$\Theta_s(0) = Q_1,\ \frac{1}{2}\frac{\partial^2 q_2}{\partial \underline{x}^2}(0,0) = Q_2 \text{ and } r(0,0) = R. \quad (3.10)$$

Local optimality will be achieved according to the following definition.

Definition 3.1 *An inverse $\mathcal{H}_\infty$-optimal feedback control law $u = \mu(y, \hat{x})$ with value function $U(x)$ is also locally $\mathcal{H}_\infty$-optimal if the quadratic term of the Taylor series expansion of $U(x)$ about the equilibrium $x = 0$ is identical to the linear $\mathcal{H}_\infty$-optimal value function $U_l(x)$, i.e., $x'\left[U_{xx}(0)\right]x \equiv U_l(x)$.*

3.2 Derivative Information Unavailable

In Section 5 we drop Assumption 3.1 by rewriting system (3.1) as

$$\begin{aligned}
\dot{x} &= Ax + \check{f}(x_1) + G_1(x_1)w + B_2 u && (3.11a)\\
y &= C_1 x + \epsilon v, && (3.11b)
\end{aligned}$$

where $v \equiv 0$ is a fictitious measurement noise and $\epsilon > 0$. The introduction of the virtual disturbance v puts system (3.11) into the desired cost-to-come form. We now strengthen Assumption 3.2:

Assumption 3.6 *There exists a constant $c_n > 0$ such that $N(x_1) := g_1(x_1)g'_1(x_1) \geq c_n > 0$ for all $x_1 \in \mathbb{R}$. In the linear case $N_0 := b_1 b'_1 \geq c_n$.*

Our local objective is near-optimality with a prescribed $\mathcal{L}_2$-gain for linear dynamics

$$\dot{x} = Ax + B_1 w_l + B_2 u_l \quad (3.12a)$$
$$y_l = C_1 x + \epsilon v_l \quad (3.12b)$$
$$\mathfrak{z}_l = C_{21}\tilde{x} + C_{22}\hat{x} + D_2 u_l. \quad (3.12c)$$

Assumption 3.7 *The pair (A, B_1) is controllable.*

Assumption 3.8 $\tilde{Q}_1 := C_{21}'C_{21} > 0$, $\tilde{Q}_2 = C_{22}'C_{22} > 0$, $R = D_2'D_2 > 0$, $C_{21}'C_{22} = 0$, $C_{21}'D_2 = 0$ *and* $C_{22}'D_2 = 0$, *where* $\tilde{Q}_1 \in \mathcal{P}_n$, $\tilde{Q}_2 \in \mathcal{P}_n$, *and* $R > 0$ *are prespecified.*

This disturbance attenuation problem is equivalent to a dynamic game with cost functional (3.3) where $\mathfrak{z}_l$ is defined by (3.12c). We assume that the desired level of attenuation is achievable: $\gamma > \gamma_\epsilon^*$, where the finite optimal linear attenuation level $\gamma_\epsilon^* > 0$ is assumed to exist for each fixed value of ϵ. Given any $Q \in \mathcal{P}_n$, suboptimality with respect to (3.4) can be achieved by selecting $\tilde{Q}_1$ and $\tilde{Q}_2$ such that

$$\tilde{x}'\tilde{Q}_1\tilde{x} + \hat{x}'\tilde{Q}_2\hat{x} \geq x'Qx. \quad (3.13)$$

Our second objective is to achieve semiglobal inverse $\mathcal{H}_\infty$-optimality with respect to a cost functional of the form (3.8) where $\gamma > \gamma_\epsilon^*$. In addition, to achieve local near-optimality, we impose conditions (3.10). By near-optimality we mean that the quadratic term of the Taylor series expansion of the value function $U(\epsilon; x)$ about the equilibrium $x = 0$ is within $O(\epsilon)$ of the linear $\mathcal{H}_\infty$-optimal value function $U_l(\epsilon; x)$ for each fixed value of x, i.e., $x'[U_{xx}(\epsilon; 0)]x - U_l(\epsilon; x) = O(\epsilon)$.

4 Design with Derivative Information

The system (3.1) is equivalently written as

$$\dot{x}_1 = a_{11}x_1 + x_2 + \breve{f}_1(x_1) + g_1(x_1)w \quad (4.1a)$$
$$\dot{x}_{\{2\}} = A_{21\{2\}}x_1 + A_{\{2\}}x_{\{2\}} + \breve{f}_{\{2\}}(x_1) + G_{1\{2\}}(x_1)w + B_{2\{2\}}u \quad (4.1b)$$
$$y = C_1 x, \quad (4.1c)$$

where the pair $(A_{\{2\}}, B_{2\{2\}})$ is controllable.

4.1 Local Design: Derivative Information

For the cost-to-come methodology we define the extended linear output

$$\bar{y}_l = \dot{x}_1 - a_{11}x_1 = x_2 + b_1 w = \hat{C}_1 x_{\{2\}} + \hat{D}_1 w$$

in which the pair $(A_{\{2\}}, \hat{C}_1)$ is observable. The robust linear observer for system (3.2) is

$$\dot{\hat{x}}_{\{2\}} = A_{21\{2\}}x_1 + A_{\{2\}}\hat{x}_{\{2\}} + \hat{\Gamma}_{1_0}\varpi_l + B_{2\{2\}}u_l, \quad (4.2)$$

where

$$\varpi_l(\bar{y}_l, \hat{x}_{\{2\}}) = N_0^{-1/2}\left(\bar{y}_l - \hat{C}_1\hat{x}_{\{2\}}\right) \quad (4.3)$$

is the equivalent disturbance, and $\hat{\Gamma}_{1_0}$ is defined by Assumption 3.5. The matrix Π_s^∞ is the solution of the filtering GARE (3.6) and $W_l = \tilde{x}_{\{2\}}'(\Pi_s^\infty)^{-1}\tilde{x}_{\{2\}}$ is the value function.

By rearranging (4.3), and expanding the extended output $\bar{y}_l$ in terms of $\dot{x}_1$, we write the observer dynamics as

$$0 = \pi_s(0, \Pi_s^\infty) \quad (4.4a)$$
$$\dot{\underline{x}} = A\underline{x} + \hat{B}_1\varpi_l + B_2 u_l, \quad (4.4b)$$

where $\hat{B}_1$ is defined by Assumption 3.5 and (4.4a) represents the filtering GARE (3.6). Hence, there exists an $\mathcal{H}_\infty$-optimal control law which minimizes

$$\underline{J}_l(u_l, \varpi_l) = \int_0^\infty \left[\underline{x}'Q_2\underline{x} + Ru^2 - \gamma^2\varpi_l'\varpi_l\right] dt.$$

The value function for this problem is $V_l = \underline{x}'P_s\underline{x}$, where $P_s \in \mathcal{P}_n$ is the solution of the control GARE (3.7) and

$$u_l = \mu_l(\underline{x}) := -R^{-1}B_2'P_s\underline{x} \quad (4.5)$$

is the optimal control.

Theorem 4.1 *Under Assumptions 3.1 - 3.5, the control law (4.5), applied to the system (3.2), in conjunction with the observer (4.2), is $\mathcal{H}_\infty$-optimal with respect to the cost functional (3.3), and is suboptimal with respect to the cost functional (3.4). Furthermore, it renders the equilibrium $(x, \hat{x}_{\{2\}}) = (0, 0)$ exponentially stable when $w_l \equiv 0$. Moreover, for all $w_l(t) \in \mathcal{L}_2$, all system signals are bounded and converge to zero as $t \to \infty$.*

Proof: The proof follows from the time derivative of

$$U_l(\underline{x}, \tilde{x}_{\{2\}}) := V_l(\underline{x}) + W_l(\tilde{x}_{\{2\}}) \quad (4.6)$$

and [11]. Moreover, if $Q_1 \in \mathcal{P}_{n-1}$ and $Q_2 \in \mathcal{P}_n$ are chosen such that (3.5) holds for a given $Q \in \mathcal{P}_n$, then it can be shown that the control law (4.5) is also suboptimal with respect to the cost functional (3.4). □

4.2 Global Design: Derivative Information

We proceed to the nonlinear design by applying the cost-to-come methodology to design a robust observer. The locally optimal design [5] is then applied to the observer dynamics augmented with the extended output dynamics to obtain a locally $\mathcal{H}_\infty$-optimal and a globally inverse $\mathcal{H}_\infty$-optimal output-feedback control law.

4.2.1 Filter Design: As dictated by the cost-to-come methodology, we define

$$\bar{y} = \dot{x}_1 - a_{11}x_1 - \breve{f}_1(x_1) = x_2 + g_1(x_1)w = \hat{C}_1 x_{\{2\}} + \hat{H}_{12}(x_1)w,$$

where $\hat{H}_{12}(x_1) = g_1(x_1)$. Without loss of generality, we set $\hat{x}_{\{2\}}(0) \equiv 0$, while $x_{\{2\}}(0)$ is unknown. The robust observer for this system is of the form

$$\dot{\hat{x}}_{\{2\}} = A_{\{2\}}\hat{x}_{\{2\}} + \varphi(x_1) + \hat{\Gamma}_1(x_1, \Pi_s)\varpi + B_{2\{2\}}u, \quad (4.7)$$

where $\varphi(x_1) := A_{21\{2\}}x_1 + \breve{f}_{\{2\}}(x_1)$, $\varpi(x_1, \bar{y}, \hat{x}_{\{2\}}) := N^{-1/2}(x_1)\left(\bar{y} - \hat{C}_1\hat{x}_{\{2\}}\right)$, and

$$\hat{\Gamma}_1 := \left(G_{1\{2\}}(x_1)g_1'(x_1) + \gamma^2\Pi_s\hat{C}_1'\right)N^{-1/2}(x_1).$$

The matrix Π_s is the solution of the differential Riccati equation

$$\begin{aligned}\dot{\Pi}_s &= \Pi_s \hat{A}' + \hat{A}\Pi_s + \gamma^{-2}\hat{B}\hat{B}' \\ &\quad -\Pi_s\left[\gamma^2 \hat{C}_1' N^{-1}(x_1)\hat{C}_1 - \Theta_s\right]\Pi_s, \quad (4.8)\end{aligned}$$

with initial condition $\Pi_s(0) = \Pi_s^\infty > 0$, where

$$\begin{aligned}\hat{A}(x_1) &:= A_{\{2\}} - G_{1\{2\}}(x_1) g_1'(x_1) N^{-1}(x_1)\hat{C}_1 \\ \hat{B}(x_1) &:= \left(G_{1\{2\}}\left[I - g_1' N^{-1} g_1\right] G_{1\{2\}}'\right)^{1/2},\end{aligned}$$

and $\hat{A}_0 \equiv \hat{A}(0)$, $\hat{B}_0 \equiv \hat{B}(0)$. Note that whenever $G_1(x_1) \equiv B_1$, the DRE (4.8) is time-invariant.

We let $\Theta_s = \gamma^{-2}\Pi_s^{-1} M \Pi_s^{-1}$ where $M = \gamma^2 \Pi_s^\infty Q_1 \Pi_s^\infty \in \mathcal{P}_{n-1}$ to ensure that the DRE (4.8) has a symmetric positive definite solution for all time. The matrix Q_1 is prescribed by the cost functional (3.3), and Π_s^∞ is the steady-state solution of the filtering GARE (3.6). This choice of Θ_s guarantees that if $x_1(t) \in \mathcal{L}_\infty$, then $\Pi_s(t)$ is symmetric positive definite for all $t \geq 0$ and there exist constant positive definite matrices which bound $\Pi_s(t)$ from above and from below (see [4]). Furthermore, $\lim_{t\to\infty} x_1(t) = 0$ implies

$$\lim_{t\to\infty} \Pi_s(t) = \Pi_s^\infty \quad \text{and} \quad \lim_{t\to\infty} \Theta_s(t) = Q_1.$$

The value function for the filtering problem is $W = \tilde{x}_{\{2\}}' \Pi_s^{-1} \tilde{x}_{\{2\}}$, and its time derivative satisfies

$$\dot{W} \leq -\left|\tilde{x}_{\{2\}}\right|^2_{\Theta_s} - \gamma^2 \varpi'\varpi + \gamma^2 w'w.$$

4.2.2 Control Design: The observer dynamics and the auxiliary output equation are rewritten as

$$\begin{aligned}\dot{\Pi}_s &= \pi_s(\underline{x}_1, \Pi_s) &(4.9a)\\ \dot{\underline{x}} &= A\underline{x} + \breve{f}(\underline{x}_1) + \hat{G}_1(\underline{x}_1, \Pi_s)\varpi + B_2 u, &(4.9b)\end{aligned}$$

where $\hat{G}_1(\underline{x}_1, \Pi_s) := \left[N^{1/2}(\underline{x}_1) \quad \hat{\Gamma}_1'(\underline{x}_1, \Pi_s)\right]'$, and ϖ is the equivalent disturbance. The linearization of this system is given by (4.4) where $\Pi_s^\infty \in \mathcal{P}_{n-1}$ is the steady-state solution of the filtering GARE (3.6), and $\hat{B}_1 := \hat{G}_1(0, \Pi_s^\infty)$. As shown in [5], the solution P_s of the GARE (3.7) admits a unique Cholesky factorization of the form $L'\Delta L$. Hence, the locally optimal design procedure of [5] can be utilized to construct a control law

$$u = \mu(\underline{x}, \Pi_s) := -r^{-1}(\underline{x}, \Pi_s) B_2' \Delta \Phi(\underline{x}, \Pi_s) \quad (4.10)$$

and a diffeomorphism $z = \Phi(\underline{x}, \Pi_s)$ for system (4.9) such that the time derivative of $V(\underline{x}, \Pi_s) = z'\Delta z$ satisfies

$$\dot{V} \leq -q_2(\underline{x}, \Pi_s) - r(\underline{x}, \Pi_s) u^2 + \gamma^2 \varpi'\varpi.$$

Theorem 4.2 *Under Assumptions 3.1 - 3.5, there exist positive functions $q_1(\tilde{x}_{\{2\}}, \Pi_s)$, $q_2(\underline{x}, \Pi_s)$, and $r(\underline{x}, \Pi_s)$, which satisfy property (3.10) and*

$$q_1(\tilde{x}_{\{2\}}, \Pi_s) \in \mathcal{P}_{\geq 0}(\tilde{x}_{\{2\}}), \quad q_2(\underline{x}, \Pi_s) \in \mathcal{P}_{\geq 0}(\underline{x}),$$

$r(\underline{x}, \Pi_s) \in \mathcal{P}_{>0}(\underline{x})$, for all $\Pi_s \in \mathcal{P}_{n-1}$, such that the control law (4.10) applied to system (4.1) with the observer (4.7) and the filtering differential Riccati equation (4.8), is locally $\mathcal{H}_\infty$-optimal with respect to the cost functional (3.3) and globally inverse $\mathcal{H}_\infty$-optimal with respect to the cost functional (3.8). In the absence of a disturbance, $w \equiv 0$, the equilibrium $(x, \hat{x}_{\{2\}}, \Pi_s) = (0, 0, \Pi_s^\infty)$ is GAS and LES. In the presence of $w(t) \in \mathcal{L}_2$, all system signals are bounded and $(x, \hat{x}_{\{2\}}, \Pi_s) \to (0, 0, \Pi_s^\infty)$ as $t \to \infty$.

Proof: With this control law, the system composed of (4.1), the observer dynamics (4.7) and the DRE (4.8), is

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1 + \hat{x}_2 + \breve{f}_1(x_1) + N^{1/2}(x_1)\varpi &(4.11a)\\ \dot{\hat{x}}_{\{2\}} &= A_{21\{2\}}x_1 + A_{\{2\}}\hat{x}_{\{2\}} + \breve{f}_{\{2\}}(x_1) \\ &\quad +\hat{\Gamma}_1(x_1, \Pi_s)\varpi + B_{2\{2\}}u &(4.11b)\\ \dot{\tilde{x}}_{\{2\}} &= A_{\{2\}}\tilde{x}_{\{2\}} + G_{1\{2\}}(x_1)w - \hat{\Gamma}_1(x_1, \Pi_s)\varpi &(4.11c)\\ \dot{\Pi}_s &= \pi_s(x_1, \Pi_s). &(4.11d)\end{aligned}$$

The value function for the extended system (4.11) is

$$U(\xi_s) := V(\underline{x}, \Pi_s) + W(\tilde{x}_{\{2\}}, \Pi_s) \geq 0, \quad (4.12)$$

where $\xi_s := \left[x_1 \quad \hat{x}_{\{2\}}' \quad \tilde{x}_{\{2\}}' \quad \vec{\Pi}_s'\right]' \equiv \left[\underline{x}' \quad \tilde{x}_{\{2\}}' \quad \vec{\Pi}_s'\right]'$, and its time derivative satisfies

$$\begin{aligned}\dot{U} &\leq -q_1(\tilde{x}_{\{2\}}, \Pi_s) - q_2(\underline{x}, \Pi_s) \\ &\quad -r(\underline{x}, \Pi_s)u^2 + \gamma^2 w'w.\end{aligned}$$

Note that $U(\xi_s)$ is positive definite and radially unbounded in $(\underline{x}(t), \tilde{x}_{\{2\}}) \in \mathbb{R}^{2n-1}$ for each $\Pi_s \in \mathcal{P}_{n-1}$, and that there exists a function $\rho(\cdot) \in \mathcal{K}_\infty$, such that $\rho(|x_1|) \leq V(\underline{x}, \Pi_s) \leq U(\xi_s)$ for all $\underline{x} \in \mathbb{R}^n$, $\hat{x}_{\{2\}} \in \mathbb{R}^{n-1}$, and all $\Pi_s \in \mathbb{R}^{n-1\times n-1}$. Although $U(\xi_s)$ is not radially unbounded in Π_s, it can be shown that $\xi_s(t)$ is defined for all $t \in [0, \infty)$ and that $x_1(t) \to 0$, $\hat{x}_{\{2\}}(t) \to 0$, and $\tilde{x}_{\{2\}}(t) \to 0$ as $t \to \infty$. Furthermore, $\Pi_s(t) \to \Pi_s^\infty$ and $\Theta_s(\Pi_s(t)) \to Q_1$ as $t \to \infty$. Moreover, at the equilibrium, $\xi_{s_0} := (0, 0_{n-1}', 0_{n-1}', \vec{\Pi}_s^{\infty\prime})$, the value function (4.12) satisfies

$$\xi_s'\left[U_{\xi_s\xi_s}(\xi_{s_0})\right]\xi_s = \tilde{x}_{\{2\}}'(\Pi_s^\infty)^{-1}\tilde{x}_{\{2\}} + \underline{x}' P_s \underline{x} \equiv U_l,$$

where U_l is the value function (4.6) for the linear problem. Hence, the control law (4.10) achieves local $\mathcal{H}_\infty$-optimality with respect to the cost functional (3.3), and, by Theorem 4.1, achieves local suboptimality with respect to the cost functional (3.4). □

5 Locally Near-Optimal Design

We now drop the preliminary Assumption 3.2 that the derivative of the output measurement is available for feedback. Instead, we follow [10, 12] and modify the system by adding a fictitious measurement noise multiplied by a known positive constant ϵ. The modified system satisfies the assumptions of the cost-to-come methodology so that a robust observer can be constructed. In the limit as $\epsilon \to 0$, this observer matches exactly the observer obtained in Section 4. With ϵ sufficiently small, this observer will help us design a system which is semiglobally inverse $\mathcal{H}_\infty$-optimal and locally near-optimal with a prescribed $\mathcal{L}_2$-gain.

5.1 Local Design

The $\mathcal{H}_\infty$-optimal filter for system (3.12) is given by

$$\begin{aligned} 0 &= \Pi^\infty(\epsilon)A' + A\Pi^\infty(\epsilon) + \gamma^{-2}B_1B_1' \\ &\quad -\Pi^\infty(\epsilon)\left[\frac{\gamma^2}{\epsilon^2}C_1'C_1 - \tilde{Q}_1\right]\Pi^\infty(\epsilon) \qquad (5.1)\\ \dot{\hat{x}} &= A\hat{x} + \frac{\gamma^2}{\epsilon}\Pi^\infty(\epsilon)C_1'\zeta + B_2u, \qquad (5.2)\end{aligned}$$

where $\zeta := \epsilon^{-1}C_1\tilde{x} \equiv \epsilon^{-1}\tilde{x}_1$. By Assumptions 3.7 and 3.8, and the fact that (A, C_1) is observable, a symmetric solution $\Pi^\infty(\epsilon) \in \mathcal{P}_n$ to the filtering GARE (5.1) exists for all $\gamma > \gamma_\epsilon$ [2]. The value function for the $\mathcal{H}_\infty$-optimal filtering problem is $W_l(\epsilon;\tilde{x}) := \tilde{x}'\left(\Pi^\infty(\epsilon)\right)^{-1}\tilde{x}$.

Similar to singularly perturbed optimal control problems and $\mathcal{H}_\infty$-optimal control problems, the solution to the GARE (5.1) admits an asymptotic expansion about $\epsilon = 0$. Thus, the filter value function can be rewritten as

$$W_l(\epsilon;\tilde{x}) = \tilde{x}_{\{2\}}'\left(\Pi_s^\infty\right)^{-1}\tilde{x}_{\{2\}} + O(\epsilon).$$

To meet the local $\mathcal{H}_\infty$-optimality objective we construct a linear $\mathcal{H}_\infty$-optimal control law $u = \mu_l(\hat{x})$ which minimizes the cost functional

$$\underline{J}_l(u_l, \zeta) = \int_0^\infty \left[\hat{x}'\tilde{Q}_2\hat{x} + Ru_l^2 - \gamma^2\zeta^2\right]dt. \qquad (5.3)$$

when applied to the observer dynamics (5.2) where ζ plays the role of the equivalent disturbance. The control GARE corresponding to this $\mathcal{H}_\infty$-optimal control problem is

$$\begin{aligned} 0 &= P(\epsilon)A + A'P(\epsilon) + \tilde{Q}_2 \\ &+P(\epsilon)\left[\frac{\gamma^2}{\epsilon^2}\Pi^\infty(\epsilon)C_1'C_1\Pi^\infty(\epsilon) - B_2R^{-1}B_2'\right]P(\epsilon). (5.4)\end{aligned}$$

By assumption, the $\mathcal{H}_\infty$-optimal control problem has a solution, and the optimal control law

$$u = \mu_l(\epsilon;\hat{x}) := -R^{-1}B_2'P(\epsilon)\hat{x} \qquad (5.5)$$

minimizes the cost functional (5.3) with value function $V_l(\epsilon;\hat{x}) = \hat{x}'P(\epsilon)\hat{x}$. Just as in the filtering case, the solution to the control GARE (5.4) is analyzed as $\epsilon \to 0$. From the dominant terms in the expansion of $\Pi^\infty(\epsilon)$, we note that if $\tilde{Q}_2 \equiv Q_2$, then the control value function is equivalently written as

$$V_l(\epsilon;\hat{x}) = \underline{x}'P_s\underline{x} + O(\epsilon).$$

Theorem 5.1 *Under Assumption 3.7 with $\gamma > \gamma_\epsilon$, the control law (5.5), applied to system (3.12) with the observer (5.1) - (5.2), is $\mathcal{H}_\infty$-optimal with respect to the cost functional (3.3), and is suboptimal with respect to the cost functional (3.4). Furthermore, it renders the equilibrium $(x, \hat{x}) = (0, 0)$ exponentially stable when $w_l \equiv 0$. Moreover, for all $w_l(t) \in \mathcal{L}_2$, all system signals are bounded and converge to zero as $t \to \infty$.*

Proof: The proof follows from the derivative of the value function $U_l(\epsilon;\tilde{x},\hat{x}) = V_l(\epsilon;\hat{x}) + W_l(\epsilon;\tilde{x})$, [11], and the fact that $U_l = \tilde{x}_{\{2\}}'\left(\Pi_s^\infty\right)^{-1}\tilde{x}_{\{2\}} + \underline{x}'P_s\underline{x} + O(\epsilon)$. □

5.2 Nonlinear Design

We now proceed to the output-feedback stabilization of the nonlinear system (3.11).

5.2.1 Filter Design: It can be shown that the worst-case filter for system (3.11) obtained through the cost-to-come methodology does not appear in strict-feedback form. As a result, we are forced to use a reduced order observer obtained in the limit as $\epsilon \to 0$, or

$$\begin{aligned} \dot{\hat{x}}_1 &= a_{11}x_1 + \hat{x}_2 + \breve{f}_1(x_1) \\ &\quad + \frac{1}{\epsilon}N^{1/2}(x_1 - \hat{x}_1) \qquad (5.6a)\\ \dot{\hat{x}}_{\{2\}} &= A_{\{2\}}\hat{x}_{\{2\}} + \varphi(x_1) + \hat{B}_2u \\ &\quad + \frac{1}{\epsilon}\hat{\Gamma}_1(x_1 - \hat{x}_1) \qquad (5.6b)\\ \dot{\Pi}_s &= \pi_s(x_1, \Pi_s), \qquad (5.6c)\end{aligned}$$

where $\hat{x}_1(0) = \hat{x}_{1_0}$, $\hat{x}_{\{2\}}(0) := 0$, $\Pi_s(0) := \Pi_s^\infty$, and $\hat{x}_{1_0}$ is the initial condition for the estimate of the measurement. If $v \equiv 0$, then we can always set $\hat{x}_{1_0} = x_1(0)$. The approximate filter (5.6) is rewritten in terms of the fast error dynamics $\zeta(t)$ as

$$\begin{aligned} \dot{\hat{x}}_{\{2\}} &= A_{\{2\}}\hat{x}_{\{2\}} + \varphi(x_1) + \hat{\Gamma}_1(x_1, \Pi_s)\zeta + \hat{B}_2u \quad (5.7)\\ \epsilon\dot{\zeta} &= -N^{1/2}\zeta + \hat{C}_1x_{\{2\}} + g_1(x_1)w - \hat{C}_1\hat{x}_{\{2\}}. \quad (5.8)\end{aligned}$$

Using singular perturbation arguments [8], it can be shown that as $\epsilon \to 0$ the approximate filter reduces to the filter (4.7) obtained in Section 4.

5.2.2 Stability and Optimality Properties: The system (3.11), the filter (5.6), the reduced DRE (4.8), and the fast error dynamics (5.8) are written together with $\xi := \begin{bmatrix} x_1 & \hat{x}_{\{2\}}' & \tilde{x}_{\{2\}}' & \vec{\Pi}_s' \end{bmatrix}'$ as

$$\begin{aligned} \dot{\xi} &= \mathcal{A}_{11}(\xi) + \mathcal{A}_{12}(\xi)\zeta + \mathcal{B}_{11}(\xi)w + \mathcal{B}_{12}(\xi)u \\ \epsilon\dot{\zeta} &= \mathcal{A}_{21}(\xi) - \mathcal{A}_{22}(\xi)\zeta + \mathcal{B}_{21}(\xi)w.\end{aligned}$$

where $\mathcal{A}_{22}(\xi) \equiv \left(\mathcal{B}_{21}\mathcal{B}_{21}'\right)^{1/2} \geq c_n^{1/2} > 0$ for all ξ. The feedback control law

$$u = -\bar{r}^{-1}(z, \Pi_s)B_2'\Delta z\big|_{z=\Phi(x_1,\hat{x}_{\{2\}},\Pi_s)} \qquad (5.9)$$

obtained in Section 4 will be utilized without any changes, except that the states $(\hat{x}_{\{2\}}, \Pi_s)$ will now be provided by filter (5.6) rather than by the filter composed of (4.7) and (4.8).

In preparation of Theorem 5.2, we define the composite state $\xi_c := [\xi' \quad \zeta]' \in \mathrm{I\!R} \times \mathrm{I\!R}^{n-1} \times \mathrm{I\!R}^{n-1} \times \mathcal{P}_n \times \mathrm{I\!R}$, and the system

$$\dot{\xi}_c = \mathcal{F}(\xi_c) + \mathcal{G}_1(\xi_c)w + \mathcal{G}_2u, \qquad (5.10)$$

where

$$\mathcal{F}(\xi_c) := \begin{bmatrix} \mathcal{A}_{11}(\xi) + \mathcal{A}_{12}(\xi)\zeta \\ \left(\mathcal{A}_{21}(\xi) - \mathcal{A}_{22}(\xi)\zeta\right)/\epsilon \end{bmatrix},$$

$$\mathcal{G}_1(\xi_c) := \begin{bmatrix} \mathcal{B}_{11}(\xi) \\ \mathcal{B}_{21}(\xi)/\epsilon \end{bmatrix}, \quad \text{and} \quad \mathcal{G}_2 := \begin{bmatrix} \mathcal{B}_{12} \\ 0 \end{bmatrix}.$$

The equilibrium is $\xi_{c_0} = \left[0'_{2n-1} \quad \vec{\Pi}_s^{\infty'} \quad 0\right]'$, and the composite Lyapunov function is defined as

$$\mathcal{V}(\xi_c) = U(\xi) + \epsilon\mathcal{U}(\xi,\zeta),$$

where

$$\mathcal{U} := \gamma^2 \left|\zeta - \mathcal{A}_{22}^{-1}\mathcal{A}_{21} - \frac{1}{2\gamma^2}\mathcal{A}_{22}^{-1}\mathcal{B}_{21}\mathcal{B}'_{11}U'_\xi\right|^2_{\mathcal{A}_{22}^{-1}}.$$

Moreover, the following sets are of interest:

$$\begin{aligned}\Omega_\xi(\beta) &:= \left\{\xi \in \mathbb{R}^{2n-1} \times \mathcal{P}_n \mid U(\xi) \le \beta\right\}\\ \Omega_\zeta(\beta,\epsilon) &:= \left\{\zeta \in \mathbb{R} \mid \epsilon\mathcal{U}(\xi,\zeta) \le \beta,\ \forall \xi \in \Omega_\xi(\beta)\right\}\\ \Omega(\beta,\epsilon) &:= \Omega_\xi(\beta) \times \Omega_\zeta(\beta,\epsilon),\end{aligned}$$

where $\beta > 0$ and $\epsilon > 0$ are known constants. The disturbance signal $w(t)$ will be permitted to be a member of the set $\mathcal{W}_2(c_{w_2}) := \{w(t) \in \mathcal{L}_2 \mid \|w(t)\|_2 \le c_{w_2}\}$, where c_{w_2} is a prespecified known positive constant. We are now ready to state the main theorem of this section (see [4] for the details of the proof).

Theorem 5.2 *Consider the nonlinear system (5.10) with Assumptions 3.4 - 3.8. Then, for each pair (β, c_{w_2}) there exists a constant ϵ_2^* such that the closed-loop system (5.10), (5.9) with $\epsilon \in (0, \epsilon_2^*)$ and*

$$\xi_c(0) \in \Omega(\beta,\epsilon) \cap \left\{\Pi_s(0) = \Pi_s^\infty,\ \hat{x}_{\{2\}}(0) = 0_{n-1}\right\}$$

and $w(t) \in \mathcal{W}_2(c_{w_2})$ achieves:

1. *Semiglobal inverse $\mathcal{H}_\infty$-optimality with respect to a cost functional of the form*

   $$\hat{J}(u,w) = \int_0^\infty \left[\mathcal{Q}(\xi_c) + r(\xi_c)u^2 - \gamma^2 w'w\right] dt,$$

   where $\Pi_s \in \mathcal{P}_{n-1}$, $\mathcal{Q}(\underline{x}, \tilde{x}_{\{2\}}, \Pi_s) \in \mathcal{P}_{\ge 0}(\underline{x}, \tilde{x}_{\{2\}})$, $r(\underline{x}, \Pi_s) \in \mathcal{P}_{>0}(\underline{x})$, and $\gamma > \gamma_\epsilon^* > 0$.

2. *The boundedness of all signals, and their convergence to the equilibrium, i.e., $\xi_c(t) \to \xi_{c_0}$ as $t \to \infty$.*

3. *Local near-optimality with respect to the cost functional (3.3) where $\tilde{Q}_1$, $\tilde{Q}_2$ and R are prespecified positive-definite matrices, and $\gamma > \gamma_\epsilon^* > 0$. Moreover, local suboptimality can be achieved with respect to the cost functional (3.4) where Q and R are prespecified positive-definite matrices.*

6 Conclusions

A dynamic $\mathcal{H}_\infty$ control law was introduced in Section 4 which obtains local $\mathcal{H}_\infty$-optimality and global inverse $\mathcal{H}_\infty$-optimality under the assumption that the derivative of the output measurement is available for feedback. When that assumption is dropped in Section 5 the control law achieves local near-optimality and semiglobal inverse $\mathcal{H}_\infty$-optimality. Unlike standard linear $\mathcal{H}_\infty$-designs which penalize the actual state of the system, the performance index we consider penalizes the estimation error and the estimate of the state. When the penalty matrices are chosen carefully, the optimal controller which corresponds to the modified performance index is shown to be suboptimal with respect to the standard $\mathcal{H}_\infty$-performance index.

References

[1] Joseph A. Ball, J. William Helton, and Michael L. Walker, "$\mathcal{H}_\infty$ control for nonlinear systems with output feedback," *IEEE Trans. on Automatic Control*, vol. 38, no. 4, pp. 546–559, April 1993.

[2] Tamer Başar and Pierre Bernhard, *$\mathcal{H}_\infty$-Optimal Control and Related Minimax Design Problems*, Birkhäuser, Boston, second ed., 1995, first ed., 1991.

[3] Gary Didinsky and Tamer Başar, "Design of minimax controllers for linear systems with non-zero initial states under specified information structures," *Int. J. of Robust and Nonlinear Control*, vol. 2, pp. 1–30, 1992.

[4] Kenan Ezal, *Disturbance Attenuating Control of Nonlinear Systems with Local Optimality*, Ph.D. thesis, University of California, Santa Barbara, 1998.

[5] Kenan Ezal, Zigang Pan, and Petar V. Kokotović, "Locally optimal backstepping design," in *Proceedings of the 36th IEEE Conference on Decision and Control*, San Diego, California, December 1997, pp. 1767–1773, and to appear in *IEEE Trans. on Automatic Control*.

[6] Randy A. Freeman and Petar V. Kokotović, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*, Birkhäuser, Boston, 1996.

[7] Alberto Isidori, "$\mathcal{H}_\infty$-control via measurement feedback for affine nonlinear systems," *Int. J. of Robust and Nonlinear Control*, vol. 4, pp. 553–574, 1994.

[8] Petar V. Kokotović, Hassan K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*, Academic Press, New York, 1986.

[9] Miroslav Krstić, Ioannis Kanellakopoulos, and Petar V. Kokotović, *Nonlinear and Adaptive Control Design*, John Wiley & Sons, New York, 1995.

[10] Zigang Pan and Tamer Başar, "Adaptive controller design for tracking and disturbance attenuation in parametric-strict-feedback nonlinear systems," *IEEE Trans. on Automatic Control*, vol. 43, no. 8, pp. 1066–1083, August 1998.

[11] Andrew Teel, "Asymptotic convergence from $\mathcal{L}_p$ stability," Technical Report CCEC 98-0331, Center for Control Engineering and Computation, University of California, Santa Barbara. Submitted to *IEEE Trans. on Automatic Control*, March 1998.

[12] I. Egemen Tezcan and Tamer Başar, "Disturbance attenuating adaptive controllers for parametric strict feedback nonlinear systems with output measurements," in *Proc. of the American Control Conference*, Albuquerque, New Mexico, June 1997, pp. 823–827, and to appear in *ASME J. of Dynamic Systems, Measurement and Control*.

[13] Arjan J. van der Schaft, "Nonlinear state space $\mathcal{H}_\infty$ control theory," in *Essays on Control: Perspectives in the Theory and its Applications*, H. L. Trentelman and J. C. Willems, Eds., pp. 153–190. Birkhäuser, 1993.